CHAPTER 6

Elementary Prime Number Theory

6.1. Euclid's Theorem Revisited

We have already seen the elegant and simple proof of Euclid's theorem, that there are infinitely many primes. Here we begin by proving a slightly stronger result.

THEOREM 6.1. *The series*

$$\sum_p \frac{1}{p}$$

is divergent.

PROOF. For every real number $X \geqslant 2$, write

$$P_X = \prod_{p \leqslant X} \left(1 - \frac{1}{p}\right)^{-1}.$$

Then

$$\log P_X = -\sum_{p \leqslant X} \log\left(1 - \frac{1}{p}\right) = S_1 + S_2,$$

where

$$S_1 = \sum_{p \leqslant X} \frac{1}{p} \quad \text{and} \quad S_2 = \sum_{p \leqslant X} \sum_{h=2}^{\infty} \frac{1}{hp^h}.$$

Since

$$0 \leqslant \sum_{h=2}^{\infty} \frac{1}{hp^h} \leqslant \sum_{h=2}^{\infty} \frac{1}{p^h} = \frac{1}{p(p-1)},$$

we have

$$0 \leqslant S_2 \leqslant \sum_p \frac{1}{p(p-1)} \leqslant \sum_{n=2}^{\infty} \frac{1}{n(n-1)} = 1,$$

so that $0 \leqslant S_2 \leqslant 1$. On the other hand, as $X \to \infty$, we have

$$P_X = \prod_{p \leqslant X} \left(\sum_{h=0}^{\infty} \frac{1}{p^h}\right) \geqslant \sum_{n \leqslant X} \frac{1}{n} \to \infty.$$

The result follows. ◯

For every real number $X \geqslant 2$, we write

$$\pi(X) = \sum_{p \leqslant X} 1,$$

so that $\pi(X)$ denotes the number of primes in the interval $[2, X]$. This function has been studied extensively by number theorists, and attempts to study it in depth have led to major developments in other important branches of mathematics.

As can be expected, many conjectures concerning the distribution of primes were made based purely on numerical evidence, including the celebrated Prime number theorem, proved in 1896 by Hadamard and de la Vallée Poussin, that

$$\lim_{X \to \infty} \frac{\pi(X) \log X}{X} = 1.$$

Here we be concerned with the weaker result of Tchebycheff, that there exist positive absolute constants c_1 and c_2 such that for every real number $X \geqslant 2$, we have

$$c_1 \frac{X}{\log X} < \pi(X) < c_2 \frac{X}{\log X}.$$

6.2. The von Mangoldt Function

The study of the function $\pi(X)$ usually involves, instead of the characteristic function of the primes, a function which counts not only primes, but prime powers as well, and with weights. Accordingly, we introduce the von Mangoldt function $\Lambda : \mathbb{N} \to \mathbb{C}$, defined for every $n \in \mathbb{N}$ by writing

$$\Lambda(n) = \begin{cases} \log p, & \text{if } n = p^u, \text{ with } p \text{ prime and } u \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

THEOREM 6.2. *For every $n \in \mathbb{N}$, we have*

$$\sum_{m \mid n} \Lambda(m) = \log n.$$

PROOF. The result is clearly true for $n = 1$, so it remains to consider the case $n \geqslant 2$. Suppose that $n = p_1^{u_1} \ldots p_r^{u_r}$ is the canonical decomposition of n. Then the only non-zero contribution to the sum on the left hand side comes from those natural numbers m of the form $m = p_j^{v_j}$ with $j = 1, \ldots, r$ and $1 \leqslant v_j \leqslant u_j$. It follows that

$$\sum_{m \mid n} \Lambda(m) = \sum_{j=1}^{r} \sum_{v_j=1}^{u_j} \log p_j = \sum_{j=1}^{r} \log p_j^{u_j} = \log n,$$

and this completes the proof. ◯

THEOREM 6.3. *As $X \to \infty$, we have*

$$\sum_{m \leqslant X} \Lambda(m) \left[\frac{X}{m}\right] = X \log X - X + O(\log X).$$

PROOF. It follows from Theorem 6.2 that

$$\sum_{n \leqslant X} \log n = \sum_{n \leqslant X} \sum_{m \mid n} \Lambda(m) = \sum_{m \leqslant X} \Lambda(m) \sum_{\substack{n \leqslant X \\ m \mid n}} 1 = \sum_{m \leqslant X} \Lambda(m) \left[\frac{X}{m}\right].$$

It therefore suffices to prove that as $X \to \infty$, we have

$$\sum_{n \leqslant X} \log n = X \log X - X + O(\log X). \tag{6.1}$$

To prove (6.1), note that $\log X$ is an increasing function of X. In particular, for every $n \in \mathbb{N}$, we have

$$\log n \leqslant \int_n^{n+1} \log u \, du,$$

so that

$$\sum_{n\leqslant X}\log n-\log(X+1)\leqslant\int_1^X\log u\,\mathrm{d}u.$$

On the other hand, for every $n\in\mathbb{N}$, we have

$$\log n\geqslant\int_{n-1}^n\log u\,\mathrm{d}u,$$

so that

$$\sum_{n\leqslant X}\log n=\sum_{2\leqslant n\leqslant X}\log n\geqslant\int_1^{[X]}\log u\,\mathrm{d}u=\int_1^X\log u\,\mathrm{d}u-\int_{[X]}^X\log u\,\mathrm{d}u\geqslant\int_1^X\log u\,\mathrm{d}u-\log X.$$

The inequality (6.1) now follows on noting that

$$\int_1^X\log u\,\mathrm{d}u=X\log X-X+1,$$

and this completes the proof. ○

6.3. Tchebycheff's Theorem

The crucial step in the proof of Tchebycheff's theorem concerns obtaining bounds on sums involving the von Mangoldt function. More precisely, we establish the following result.

THEOREM 6.4. *There exist positive absolute constants* c_3 *and* c_4 *such that*

$$\sum_{m\leqslant X}\Lambda(m)\geqslant\frac{1}{2}X\log 2\quad\text{if } X\geqslant c_3\tag{6.2}$$

and

$$\sum_{\frac{X}{2}<m\leqslant X}\Lambda(m)\leqslant c_4X\quad\text{if } X\geqslant 0.\tag{6.3}$$

PROOF. If $m\in\mathbb{N}$ satisfies $X/2<m\leqslant X$, then clearly $[X/2m]=0$. It follows from this and Theorem 6.3 that as $X\to\infty$, we have

$$\sum_{m\leqslant X}\Lambda(m)\left(\left[\frac{X}{m}\right]-2\left[\frac{X}{2m}\right]\right)=\sum_{m\leqslant X}\Lambda(m)\left[\frac{X}{m}\right]-2\sum_{m\leqslant\frac{X}{2}}\Lambda(m)\left[\frac{X}{2m}\right]$$

$$=(X\log X-X+O(\log X))-2\left(\frac{X}{2}\log\frac{X}{2}-\frac{X}{2}+O(\log X)\right)=X\log 2+O(\log X).$$

Hence there exists a positive absolute constant c_5 such that for all sufficiently large X, we have

$$\frac{1}{2}X\log 2<\sum_{m\leqslant X}\Lambda(m)\left(\left[\frac{X}{m}\right]-2\left[\frac{X}{2m}\right]\right)<c_5X.$$

We now consider the function $[\alpha]-2[\alpha/2]$. Clearly $[\alpha]-2[\alpha/2]<\alpha-2(\alpha/2-1)=2$. Note that the left hand side is an integer, so we must have $[\alpha]-2[\alpha/2]\leqslant 1$. It follows that for all sufficiently large X, we have

$$\frac{1}{2}X\log 2<\sum_{m\leqslant X}\Lambda(m).$$

The inequality (6.2) follows. On the other hand, if $X/2<m\leqslant X$, then $[X/m]=1$ and $[X/2m]=0$, so that for all sufficiently large X, we have

$$\sum_{\frac{X}{2}<m\leqslant X}\Lambda(m)\leqslant c_5X.$$

The inequality (6.3) follows easily. ○

We now state and prove Tchebycheff's theorem.

THEOREM 6.5 (Tchebycheff). *There exist positive absolute constants c_1 and c_2 such that for every real number $X \geqslant 2$, we have*

$$c_1 \frac{X}{\log X} < \pi(X) < c_2 \frac{X}{\log X}.$$

PROOF. To prove the lower bound, note that

$$\sum_{m \leqslant X} \Lambda(m) = \sum_{\substack{p,n \\ p^n \leqslant X}} \log p = \sum_{p \leqslant X} (\log p) \sum_{1 \leqslant n \leqslant \left[\frac{\log X}{\log p}\right]} 1 = \sum_{p \leqslant X} (\log p) \left[\frac{\log X}{\log p}\right] \leqslant \pi(X) \log X.$$

It follows from (6.2) that

$$\pi(X) \geqslant \frac{X \log 2}{2 \log X} \quad \text{if } X \geqslant c_3.$$

Since $\pi(2) = 1$, we get the lower bound for a suitable choice of c_1.

To prove the upper bound, note that in view of (6.3) and the definition of the von Mangoldt function, the inequality

$$\sum_{\frac{X}{2^{j+1}} < p \leqslant \frac{X}{2^j}} \log p \leqslant c_4 \frac{X}{2^j}$$

holds for every integer $j \geqslant 0$ and every real number $X \geqslant 0$. Suppose that $X \geqslant 2$. Let the integer $k \geqslant 0$ be defined such that $2^k < X^{\frac{1}{2}} \leqslant 2^{k+1}$. Then

$$\sum_{X^{\frac{1}{2}} < p \leqslant X} \log p \leqslant \sum_{j=0}^{k} \sum_{\frac{X}{2^{j+1}} < p \leqslant \frac{X}{2^j}} \log p \leqslant c_4 X \sum_{j=0}^{k} 2^{-j} < 2c_4 X,$$

so that

$$\sum_{X^{\frac{1}{2}} < p \leqslant X} 1 \leqslant \sum_{X^{\frac{1}{2}} < p \leqslant X} \frac{\log p}{\log X^{\frac{1}{2}}} < \frac{4c_4 X}{\log X},$$

whence

$$\pi(X) \leqslant X^{\frac{1}{2}} + \frac{4c_4 X}{\log X} < \frac{c_2 X}{\log X}$$

for a suitable constant c_2. ○

6.4. Some Results of Mertens

We conclude this chapter by obtaining an improvement of Theorem 6.1.

THEOREM 6.6 (Mertens). *As $X \to \infty$, we have*

$$\sum_{m \leqslant X} \frac{\Lambda(m)}{m} = \log X + O(1), \tag{6.4}$$

$$\sum_{p \leqslant X} \frac{\log p}{p} = \log X + O(1), \tag{6.5}$$

and

$$\sum_{p \leqslant X} \frac{1}{p} = \log \log X + O(1). \tag{6.6}$$

PROOF. Recall Theorem 6.3. As $X \to \infty$, we have

$$\sum_{m \leqslant X} \Lambda(m) \left[\frac{X}{m}\right] = X \log X - X + O(\log X).$$

Clearly $[X/m] = X/m + O(1)$, so that as $X \to \infty$, we have

$$\sum_{m \leqslant X} \Lambda(m) \left[\frac{X}{m}\right] = X \sum_{m \leqslant X} \frac{\Lambda(m)}{m} + O\left(\sum_{m \leqslant X} \Lambda(m)\right).$$

It follows from (6.3) that

$$\sum_{m\leqslant X}\Lambda(m)\leqslant\sum_{j=0}^{\infty}\sum_{\frac{X}{2^{j+1}}<m\leqslant\frac{X}{2^j}}\Lambda(m)\leqslant 2c_4X,$$

so that as $X\to\infty$, we have

$$X\sum_{m\leqslant X}\frac{\Lambda(m)}{m}=X\log X+O(X).$$

The inequality (6.4) follows. Next, note that

$$\sum_{m\leqslant X}\frac{\Lambda(m)}{m}=\sum_{\substack{p,k\\p^k\leqslant X}}\frac{\log p}{p^k}=\sum_{p\leqslant X}\frac{\log p}{p}+\sum_{p\leqslant X}(\log p)\sum_{2\leqslant k\leqslant\frac{\log X}{\log p}}\frac{1}{p^k}.$$

As $X\to\infty$, we have

$$\sum_{p\leqslant X}(\log p)\sum_{2\leqslant k\leqslant\frac{\log X}{\log p}}\frac{1}{p^k}\leqslant\sum_{p\leqslant X}(\log p)\sum_{k=2}^{\infty}\frac{1}{p^k}=\sum_{p\leqslant X}\frac{\log p}{p(p-1)}\leqslant\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)}=O(1).$$

The inequality (6.5) follows. Finally, for every real number $X\geqslant 2$, let

$$T(X)=\sum_{p\leqslant X}\frac{\log p}{p}.$$

It follows from (6.5) that there exists a positive absolute constant c_6 such that $|T(X)-\log X|<c_6$ whenever $X\geqslant 2$. On the other hand,

$$\begin{aligned}\sum_{p\leqslant X}\frac{1}{p}&=\sum_{p\leqslant X}\frac{\log p}{p}\left(\frac{1}{\log X}+\int_p^X\frac{1}{y\log^2 y}\,\mathrm{d}y\right)=\frac{T(X)}{\log X}+\int_2^X\frac{T(y)}{y\log^2 y}\,\mathrm{d}y\\&=\frac{T(X)-\log X}{\log X}+\int_2^X\frac{T(y)-\log y}{y\log^2 y}\,\mathrm{d}y+1+\int_2^X\frac{1}{y\log y}\,\mathrm{d}y.\end{aligned}$$

It follows that as $X\to\infty$, we have

$$\left|\sum_{p\leqslant X}\frac{1}{p}-\log\log X\right|<\frac{c_6}{\log X}+\int_2^X\frac{c_6}{y\log^2 y}\,\mathrm{d}y+1-\log\log 2=O(1).$$

The inequality (6.6) follows. ◯