FUNDAMENTALS OF ANALYSIS

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Chapter 1

THE NUMBER SYSTEM

1.1. The Real Numbers

In this chapter, we shall make a detailed study of some of the important properties of the real numbers. Most readers will be familiar with some of these properties, or have at least used most of them, perhaps sometimes unaware of their generality. Throughout, we denote the set of all real numbers by $\mathbb{R}$, and write $a \in \mathbb{R}$ to indicate that a is a real number.

We shall take an axiomatic approach to the real numbers. In other words, we offer no proof of these properties, and simply treat and accept them as given.

The first collection of properties of $\mathbb{R}$ is generally known as the Field axioms. They enable us to study arithmetic.

FIELD AXIOMS.
(A1) *For every* $a, b \in \mathbb{R}$, *we have* $a + b \in \mathbb{R}$.
(A2) *For every* $a, b, c \in \mathbb{R}$, *we have* $a + (b + c) = (a + b) + c$.
(A3) *For every* $a \in \mathbb{R}$, *we have* $a + 0 = a$.
(A4) *For every* $a \in \mathbb{R}$, *there exists* $-a \in \mathbb{R}$ *such that* $a + (-a) = 0$.
(A5) *For every* $a, b \in \mathbb{R}$, *we have* $a + b = b + a$.
(M1) *For every* $a, b \in \mathbb{R}$, *we have* $ab \in \mathbb{R}$.
(M2) *For every* $a, b, c \in \mathbb{R}$, *we have* $a(bc) = (ab)c$.
(M3) *For every* $a \in \mathbb{R}$, *we have* $a1 = a$.
(M4) *For every* $a \in \mathbb{R}$ *such that* $a \neq 0$, *there exists* $a^{-1} \in \mathbb{R}$ *such that* $aa^{-1} = 1$.
(M5) *For every* $a, b \in \mathbb{R}$, *we have* $ab = ba$.
(D) *For every* $a, b, c \in \mathbb{R}$, *we have* $a(b + c) = ab + ac$.

Remark. The properties (A1)–(A5) concern the operation addition, while the properties (M1)–(M5) concern the operation multiplication. In the terminology of group theory, we say that the set $\mathbb{R}$ forms an abelian group under addition, and that the set of all non-zero real numbers forms an abelian group under multiplication. We also say that the set $\mathbb{R}$ forms a field under addition and multiplication. The property (D) is called the Distributive law.

The second collection of properties of $\mathbb{R}$ is generally known as the Order axioms. They enable us to study inequalities.

ORDER AXIOMS.

(O1) For every $a, b \in \mathbb{R}$, exactly one of $a < b$, $a = b$, $a > b$ holds.
(O2) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $b > c$, we have $a > c$.
(O3) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$, we have $a + c > b + c$.
(O4) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $c > 0$, we have $ac > bc$.

Remark. Clearly the Order axioms as given do not appear to include many other properties of the real numbers. However, these can be deduced from the Field axioms and Order axioms.

Example 1.1.1. Suppose that the real number $a > 0$. Then the real number $-a < 0$. To see this, note first that by Axiom (A4), there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0$. Hence

$$\begin{aligned} 0 &= a + (-a) && \text{from above,} \\ &> 0 + (-a) && \text{by Axiom (O3),} \\ &= (-a) + 0 && \text{by Axiom (A5),} \\ &= -a && \text{by Axiom (A3),} \end{aligned}$$

as required.

Example 1.1.2. For every $a \in \mathbb{R}$, we have $a0 = 0$. To see this, note first that $a0 \in \mathbb{R}$, in view of Axiom (M1). On the other hand, it follows from Axioms (A3) and (D) that $a0 = a(0+0) = a0 + a0$. Note next that $-(a0) \in \mathbb{R}$ and $a0 + (-(a0)) = 0$, in view of Axiom (A4). Hence

$$\begin{aligned} 0 &= a0 + (-(a0)) && \text{from above,} \\ &= (a0 + a0) + (-(a0)) && \text{from above,} \\ &= a0 + (a0 + (-(a0))) && \text{by Axiom (A2),} \\ &= a0 + 0 && \text{by Axiom (A4),} \\ &= a0 && \text{by Axiom (A3),} \end{aligned}$$

as required.

Example 1.1.3. Suppose that the real number $a > 0$. Then the real number $a^{-1} > 0$. To see this, note first that by Axiom (M4), there exists $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$. Suppose on the contrary that it is not true that $a^{-1} > 0$. Then it follows from Axiom (O1) that $a^{-1} = 0$ or $a^{-1} < 0$. If $a^{-1} = 0$, then

$$\begin{aligned} 1 &= aa^{-1} && \text{by Axiom (M4),} \\ &= a0 \\ &= 0 && \text{by Example 1.1.2,} \end{aligned}$$

and so

$$\begin{aligned} a &= a1 && \text{by Axiom (M3),} \\ &= a0 && \text{from above,} \\ &= 0 && \text{by Example 1.1.2,} \end{aligned}$$

a contradiction. If $a^{-1} < 0$, then

$$\begin{aligned} 0 &= a0 && \text{by Example 1.1.2,} \\ &= 0a && \text{by Axiom (M5),} \\ &> a^{-1}a && \text{by Axiom (O4),} \\ &= aa^{-1} && \text{by Axiom (M5),} \\ &= 1 && \text{by Axiom (M4),} \end{aligned}$$

and so

$$\begin{aligned} 0 &= a0 && \text{by Example 1.1.2,} \\ &> a1 && \text{from above,} \\ &= a && \text{by Axiom (M3),} \end{aligned}$$

again a contradiction.

EXAMPLE 1.1.4. Suppose that the real numbers $a > 0$ and $b > 0$. Then the real number $ab > 0$. To see this, note first that by Axiom (M1), we have $ab \in \mathbb{R}$. Suppose on the contrary that it is not true that $ab > 0$. Then it follows from Axiom (O1) that $ab = 0$ or $ab < 0$. Since $b > 0$, it follows from Axiom (O1) that $b \neq 0$, from Axiom (M4) that $b^{-1} \in \mathbb{R}$, and from Example 1.1.3 that $b^{-1} > 0$. If $ab = 0$, then

$$\begin{aligned} a &= a1 && \text{by Axiom (M3),} \\ &= a(bb^{-1}) && \text{by Axiom (M4),} \\ &= (ab)b^{-1} && \text{by Axiom (M2),} \\ &= 0b^{-1} && \\ &= b^{-1}0 && \text{by Axiom (M5),} \\ &= 0 && \text{by Example 1.1.2,} \end{aligned}$$

a contradiction. If $ab < 0$, then

$$\begin{aligned} a &= a1 && \text{by Axiom (M3),} \\ &= a(bb^{-1}) && \text{by Axiom (M4),} \\ &= (ab)b^{-1} && \text{by Axiom (M2),} \\ &< 0b^{-1} && \text{by Axiom (O4),} \\ &= b^{-1}0 && \text{by Axiom (M5),} \\ &= 0 && \text{by Example 1.1.2,} \end{aligned}$$

again a contradiction.

EXAMPLE 1.1.5. Suppose that $a, b \in \mathbb{R}$ and $0 < a < b$. Then $b^{-1} < a^{-1}$. To see this, note first from Example 1.1.3 that $a^{-1} > 0$ and $b^{-1} > 0$, and from Example 1.1.4 that $b^{-1}a^{-1} > 0$. Hence

$$\begin{aligned} b^{-1} &= b^{-1}1 && \text{by Axiom (M3),} \\ &= b^{-1}(aa^{-1}) && \text{by Axiom (M4),} \\ &= b^{-1}(a^{-1}a) && \text{by Axiom (M5),} \\ &= (b^{-1}a^{-1})a && \text{by Axiom (M2),} \\ &< (b^{-1}a^{-1})b && \text{by Axiom (O4),} \\ &= (a^{-1}b^{-1})b && \text{by Axiom (M5),} \\ &= a^{-1}(b^{-1}b) && \text{by Axiom (M2),} \\ &= a^{-1}(bb^{-1}) && \text{by Axiom (M5),} \\ &= a^{-1}1 && \text{by Axiom (M4),} \\ &= a^{-1} && \text{by Axiom (M3),} \end{aligned}$$

as required.

An important subset of the set $\mathbb{R}$ of all real numbers is the set of all natural numbers, given by

$$\mathbb{N} = \{1, 2, 3, \ldots\}.$$

However, this definition does not bring out some of the main properties of the set $\mathbb{N}$ in a natural way. The following more complicated definition is therefore sometimes preferred.

AXIOMS OF THE NATURAL NUMBERS.

(N1) $1 \in \mathbb{N}$.
(N2) If $n \in \mathbb{N}$, then the number $n+1$, called the successor of n, also belongs to $\mathbb{N}$.
(N3) Every $n \in \mathbb{N}$ other than 1 is the successor of some number in $\mathbb{N}$.
(WO) Every non-empty subset of $\mathbb{N}$ has a least element.

Remark. The condition (WO) is called the Well-ordering principle.

To explain the significance of each of these four axioms, note first that Axioms (N1) and (N2) together imply that $\mathbb{N}$ contains $1, 2, 3, \ldots$. However, these two axioms alone are insufficient to exclude from $\mathbb{N}$ numbers such as 5.5. Now, if $\mathbb{N}$ contained 5.5, then by Axiom (N3), $\mathbb{N}$ must also contain $4.5, 3.5, 2.5, 1.5, 0.5, -0.5, -1.5, -2.5, \ldots$, and so would not have a least element. We therefore exclude this possibility by stipulating that $\mathbb{N}$ has a least element. This is achieved by Axiom (WO).

It can be shown that Axiom (WO) implies the Principle of induction. The following two forms of the Principle of induction are particularly useful. In fact, both are equivalent to Axiom (WO).

PRINCIPLE OF INDUCTION (WEAK FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIW1) $p(1)$ is true; and
(PIW2) $p(n+1)$ is true whenever $p(n)$ is true.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

PRINCIPLE OF INDUCTION (STRONG FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIS1) $p(1)$ is true; and
(PIS2) $p(n+1)$ is true whenever $p(m)$ is true for all $m \le n$.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

Proof of the equivalence of the Well-ordering principle and the two Principles of induction. Our first step is to show that Axiom (WO) is equivalent to the Principle of induction (strong form) (PIS).

((WO) $\Rightarrow$ (PIS)) Suppose that the conclusion of (PIS) does not hold. Then the subset

$$S = \{n \in \mathbb{N} : p(n) \text{ is false}\}$$

of $\mathbb{N}$ is non-empty. By Axiom (WO), S has a least element, n_0 say. If $n_0 = 1$, then clearly (PIS1) does not hold. If $n_0 > 1$, then $p(m)$ is true for all $m \le n_0 - 1$ but $p(n_0)$ is false, contradicting (PIS2).

((PIS) $\Rightarrow$ (WO)) Suppose that a non-empty subset S of $\mathbb{N}$ does not have a least element. Consider the statement $p(n)$, given by $n \notin S$. Then $p(1)$ is true, otherwise 1 would be the least element of S. Suppose next that $p(m)$ is true for every natural number $m \le n$, so that none of the numbers $1, 2, 3, \ldots, n$ belongs to S. Then $p(n+1)$ must also be true, for otherwise $n+1$ would be the least element of S. It now follows from (PIS) that S does not contain any element of $\mathbb{N}$, contradicting the assumption that S is a non-empty subset of $\mathbb{N}$.

Next, we complete the proof by showing that the Principle of induction (weak form) (PIW) is equivalent to the Principle of induction (strong form) (PIS).

((PIS) $\Rightarrow$ (PIW)) Suppose that (PIW1) and (PIW2) both hold. Then clearly (PIS1) holds, since it is the same as (PIW1). On the other hand, if $p(m)$ is true for all $m \le n$, then $p(n)$ is true in particular, so it follows from (PIW2) that $p(n+1)$ is true, and this gives (PIS2). It now follows from (PIS) that $p(n)$ is true for every $n \in \mathbb{N}$.

((PIW) $\Rightarrow$ (PIS)) Suppose that (PIS1) and (PIS2) both hold for a statement $p(.)$. Consider a statement $q(.)$, where $q(n)$ denotes the statement

$$p(m) \text{ is true for every } m \le n.$$

Then the two conditions (PIS1) and (PIS2) for the statement $p(.)$ imply respectively the two conditions (PIW1) and (PIW2) for the statement $q(.)$. It follows from (PIW) that $q(n)$ is true for every $n \in \mathbb{N}$, and this clearly implies that $p(n)$ is true for every $n \in \mathbb{N}$. ○

1.2. Completeness of the Real Numbers

The set $\mathbb{Z}$ of all integers is an extension of the set $\mathbb{N}$ of all natural numbers to include 0 and all numbers of the form $-n$, where $n \in \mathbb{N}$. The set $\mathbb{Q}$ of all rational numbers is the set of all real numbers of the form pq^{-1}, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. It is easy see that the Field axioms and Order axioms hold good if the set $\mathbb{R}$ is replaced by the set $\mathbb{Q}$. We therefore need to find a property that distinguishes $\mathbb{R}$ from $\mathbb{Q}$. A good starting point is the following well known result.

THEOREM 1A. *No rational number $x \in \mathbb{Q}$ satisfies $x^2 = 2$.*

Proof. Suppose that pq^{-1} has square 2, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We may assume, without loss of generality, that p and q have no common factors apart from ± 1. Then $p^2 = 2q^2$ is even, so that p is even. We can write $p = 2r$, where $r \in \mathbb{Z}$. Then $q^2 = 2r^2$ is even, so that q is even, contradicting that assumption that p and q have no common factors apart from ± 1. ○

It follows that the real number we know as $\sqrt{2}$ does not belong to the set $\mathbb{Q}$. We say that the set $\mathbb{Q}$ is not complete. Our idea is then to distinguish the set $\mathbb{R}$ from the set $\mathbb{Q}$ by completeness. In particular, we want to ensure that the set $\mathbb{R}$ contains numbers like $\sqrt{2}$.

There are a number of ways to describe the completeness of the set $\mathbb{R}$. We shall first of all introduce completeness via the Axiom of bound.

Definition. A non-empty set S of real numbers is said to be bounded above if there exists a number $K \in \mathbb{R}$ such that $x \le K$ for every $x \in S$. The number K is called an upper bound of the set S.

Definition. A non-empty set T of real numbers is said to be bounded below if there exists a number $k \in \mathbb{R}$ such that $x \ge k$ for every $x \in T$. The number k is called a lower bound of the set T.

AXIOM OF BOUND. *Suppose that a non-empty set S of real numbers is bounded above. Then there is a real number $M \in \mathbb{R}$ satisfying the following two conditions:*
(S1) For every $x \in S$, the inequality $x \le M$ holds.
(S2) For every $\epsilon > 0$, there exists $x \in S$ such that $x > M - \epsilon$.

Remark. It is not difficult to prove that the number M above is unique. It is also easy to deduce that if a non-empty set T of real numbers is bounded below, then there is a unique real number $m \in \mathbb{R}$ satisfying the following two conditions:
(I1) For every $x \in T$, the inequality $x \ge m$ holds.
(I2) For every $\epsilon > 0$, there exists $x \in T$ such that $x < m + \epsilon$.

Definition. The real number M satisfying conditions (S1) and (S2) is called the supremum of the non-empty set S, and denoted by $M = \sup S$. The real number m satisfying conditions (I1) and (I2) is called the infimum of the non-empty set S, and denoted by $m = \inf S$.

Remark. Note that the most important point of the Axiom of bound is that the supremum M is a real number. Similarly, the infimum m is also a real number.

Let us now try to understand how numbers like $\sqrt{2}$ fit into this setting. Recall that there is no rational number which satisfies the equation $x^2 = 2$. This means that the number that we know as $\sqrt{2}$ is not a rational number. We now want to show that it is a real number. Let

$$S = \{x \in \mathbb{R} : x^2 < 2\}.$$

Clearly the set S is non-empty, since $0 \in S$. On the other hand, the set S is bounded above; for example, it is not difficult to show that if $x \in S$, then we must have $x \le 2$; for if $x > 2$, then we must have $x^2 > 4$, so that $x \notin S$. Hence S is a non-empty set of real numbers and S is bounded above. It follows from the Axiom of bound that there is a real number M satisfying conditions (S1) and (S2). We shall show that $M^2 = 2$.

Suppose on the contrary that $M^2 \ne 2$. Then it follows from Axiom (O1) that $M^2 < 2$ or $M^2 > 2$. Let us investigate these two cases separately.

If $M^2 < 2$, then we have

$$(M + \epsilon)^2 = M^2 + 2M\epsilon + \epsilon^2 < 2 \qquad \text{whenever } \epsilon < \min\left\{1, \frac{2 - M^2}{2M + 1}\right\}.$$

This means that $M + \epsilon \in S$, contradicting conndition (S1).

If $M^2 > 2$, then we have

$$(M - \epsilon)^2 = M^2 - 2M\epsilon + \epsilon^2 > 2 \qquad \text{whenever } \epsilon < \frac{M^2 - 2}{2M}.$$

This implies that any $x > M - \epsilon$ will not belong to S, contradicting condition (S2).

Note that $M^2 = 2$ and M is a real number. It follows that what we know as $\sqrt{2}$ is a real number.

Example 1.2.1. The set $\mathbb{N}$ is not bounded above but is bounded below with infimum 1.

Example 1.2.2. The set $\mathbb{Z}$ is not bounded above or below.

Example 1.2.3. The closed interval $[\sqrt{2}, 2] = \{x \in \mathbb{R} : \sqrt{2} \le x \le 2\}$ is bounded above and below, with supremum 2 and infimum $\sqrt{2}$. Note that the supremum and infimum belong to the interval.

Example 1.2.4. The open interval $(\sqrt{2}, 2) = \{x \in \mathbb{R} : \sqrt{2} < x < 2\}$ is bounded above and below, with supremum 2 and infimum $\sqrt{2}$. Note that the supremum and infimum do not belong to the interval.

Example 1.2.5. The set $\{x \in \mathbb{R} : x = (-1)^n n^{-1} \text{ for some } n \in \mathbb{N}\}$ is bounded above and below, with supremum $1/2$ and infimum -1.

Example 1.2.6. The set $\{x \in \mathbb{Q} : x^2 < 2\}$ is bounded above and below, with supremum $\sqrt{2}$ and infimum $-\sqrt{2}$.

The argument concerning $\sqrt{2}$ can be adapted to prove the following result.

THEOREM 1B. *Suppose that a real number $c \in \mathbb{R}$ is positive. Then for every natural number $q \in \mathbb{N}$, there exists a unique positive real number $x \in \mathbb{R}$ such that $x^q = c$.*

We denote by $c^{1/q}$ or $\sqrt[q]{c}$ the unique positive real solution of the equation $x^q = c$ given by Theorem 1B. For every $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, we define $c^{p/q} = (c^{1/q})^p$. It can be shown that the definition of c^m, where $m = p/q$ with $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, is independent of the choice of p and q. Furthermore, the Index laws are satisfied: For every positive real number $c \in \mathbb{R}$ and rational numbers $m, n \in \mathbb{Q}$, we have $c^m c^n = c^{m+n}$ and $(c^m)^n = c^{mn}$.

We next elaborate on Example 1.2.1, and prove formally that the set $\mathbb{N}$ is not bounded above. This is a consequence of the Axiom of bound.

THEOREM 1C. (ARCHIMEDEAN PROPERTY) *For every real number $x \in \mathbb{R}$, there exists a natural number $n \in \mathbb{N}$ such that $n > x$.*

PROOF. Suppose that $x \in \mathbb{R}$, and suppose on the contrary that $n \leq x$ for every $n \in \mathbb{N}$. Then the set $\mathbb{N}$ is bounded above by x, and so has a supremum M, say. In particular, we have

$$M \geq 2, \qquad M \geq 3, \qquad M \geq 4, \qquad \ldots,$$

and so

$$M - 1 \geq 1, \qquad M - 1 \geq 2, \qquad M - 1 \geq 3, \qquad \ldots.$$

Hence $M - 1$ is an upper bound for $\mathbb{N}$, contradicting the hypothesis that M is the supremum of $\mathbb{N}$. ◯

We now establish the following important result central to the theory of mathematical analysis.

THEOREM 1D. *The rational numbers and irrational numbers are dense in the set $\mathbb{R}$. More precisely, between any two distinct real numbers, there exist a rational number and an irrational number.*

PROOF. Suppose that $x, y \in \mathbb{R}$ and $x < y$. We shall first show that there exists $r \in \mathbb{Q}$ such that $x < r < y$. The idea is very simple. Heuristically, if we choose a natural number q large enough, then the interval (qx, qy) has length greater than 1 and must contain an integer p, so that $qx < p < qy$. The formal argument is somewhat more complicated, but is based entirely on this idea.

Consider the special case when $x > 0$. By the Archimedean property, there exists $q \in \mathbb{N}$ such that $q > 1/(y-x)$, so that $1 < q(y-x)$. Consider the positive real number qx. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > qx$. Using the Well-ordering principle, let p be the smallest such natural number n. Then clearly $p - 1 \leq qx$. To see this, note that if $p = 1$, then $p - 1 = 0 < qx$; if $p \neq 1$, then $p - 1 > qx$ would contradict the definition of p. It now follows that

$$qx < p = (p-1) + 1 < qx + q(y-x) = qy, \qquad \text{so that} \qquad x < \frac{p}{q} < y.$$

Suppose now that $x \leq 0$. By the Archimedean property, there exists $k \in \mathbb{N}$ such that $k > -x$, so that $k + x > 0$. There exists $s \in \mathbb{Q}$ such that $x + k < s < y + k$, so that $x < s - k < y$. Clearly $s - k \in \mathbb{Q}$.

To show that there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$, we first use our earlier argument twice, and conclude that there exist $r_1, r_2 \in \mathbb{Q}$ such that $x < r_1 < r_2 < y$. The number

$$z = r_1 + \frac{1}{\sqrt{2}}(r_2 - r_1)$$

is clearly irrational and satisfies $r_1 < z < r_2$, and so $x < z < y$. ◯.

1.3. The Complex Numbers

In this section, we briefly review some important properties of the complex numbers. It is easy to see that the equation $x^2+1=0$ has no solution $x \in \mathbb{R}$. In order to solve this equation, we have to introduce extra numbers into our number system.

Define the number i by $\mathrm{i}^2+1=0$. We then extend the field of all real numbers by adjoining the number i, which is then combined with the real numbers by the operations addition and multiplication in accordance with the Field axioms in Section 1.1. The numbers $a+b\mathrm{i}$, where $a,b \in \mathbb{R}$, of the extended field are then added and multiplied in accordance with the Field axioms, suitably extended, and the restriction $\mathrm{i}^2+1=0$. Note that the number $a+0\mathrm{i}$, where $a \in \mathbb{R}$, behaves like the real number a.

The set $\mathbb{C} = \{z = x+y\mathrm{i} : x, y \in \mathbb{R}\}$ is called the set of all complex numbers. Note that in $\mathbb{C}$, we lose the Order axioms and the Axiom of bound.

Suppose that $z = x+y\mathrm{i}$, where $x, y \in \mathbb{R}$. The real number x is called the real part of z, and denoted by $x = \mathfrak{Re} z$. The real number y is called the imaginary part of z, and denoted by $y = \mathfrak{Im} z$. Furthermore, we write

$$|z| = \sqrt{x^2+y^2}$$

and call this the modulus of z.

Definition. A set S of complex numbers is said to be bounded if there exists a number $K \in \mathbb{R}$ such that $|z| \le K$ for every $z \in T$.

THEOREM 1E. *For every $z, w \in \mathbb{C}$, we have*
(a) $|zw| = |z||w|$; and
(b) $|z+w| \le |z|+|w|$.

Proof. The first part is left as an exercise. To prove the Triangle inequality (b), note that the result is trivial if $z+w=0$. Suppose now that $z+w \ne 0$. Then

$$\begin{aligned}\frac{|z|+|w|}{|z+w|} &= \frac{|z|}{|z+w|} + \frac{|w|}{|z+w|} = \left|\frac{z}{z+w}\right| + \left|\frac{w}{z+w}\right| \\ &\ge \mathfrak{Re}\frac{z}{z+w} + \mathfrak{Re}\frac{w}{z+w} = \mathfrak{Re}\left(\frac{z}{z+w} + \frac{w}{z+w}\right) = \mathfrak{Re} 1 = 1.\end{aligned}$$

The result follows immediately. ○

Applying the Triangle inequality a finite number of times, we can show that for every $z_1, \ldots, z_k \in \mathbb{C}$, we have

$$|z_1 + \ldots + z_k| \le |z_1| + \ldots + |z_k|.$$

We shall use this to establish the following result which shows that a polynomial is eventually dominated by its term of highest order.

THEOREM 1F. *Consider a polynomial $P(z) = a_0 + a_1 z + \ldots + a_n z^n$ in the complex variable $z \in \mathbb{C}$, with coefficients $a_0, a_1, \ldots, a_n \in \mathbb{C}$ and $a_n \ne 0$. For every $z \in \mathbb{C}$ satisfying*

$$|z_0| \ge R_0 = \frac{2(|a_0|+|a_1|+\ldots+|a_n|)}{|a_n|},$$

we have

$$\tfrac{1}{2}|a_n||z|^n \le |P(z)| \le \tfrac{3}{2}|a_n||z|^n.$$

PROOF. Note first of all that

$$|P(z)| \le |a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}| + |a_n||z|^n$$

and

$$|a_n||z|^n = |P(z) - (a_0 + a_1 z + \ldots + a_{n-1} z^{n-1})| \le |P(z)| + |a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}|.$$

It therefore remains to establish the inequality

$$|a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}| \le \tfrac{1}{2}|a_n||z|^n.$$

Clearly $R_0 > 1$, so that if $|z| \ge R_0$, we have

$$\begin{aligned}|a_0 + a_1 z + \ldots + a_{n-1} z^{n-1}| &\le |a_0| + |a_1||z| + \ldots + |a_{n-1}||z|^{n-1} \le (|a_0| + |a_1| + \ldots + |a_{n-1}|)|z|^{n-1} \\ &\le (|a_0| + |a_1| + \ldots + |a_{n-1}| + |a_n|)|z|^{n-1} = \tfrac{1}{2}R_0|a_n||z|^{n-1} \le \tfrac{1}{2}|a_n||z|^n\end{aligned}$$

as required. ○

1.4. Countability

In this brief account, we treat intuitively the distinction between finite and infinite sets. A set is finite if it contains a finite number of elements. To treat infinite sets, our starting point is the set $\mathbb{N}$ of all natural numbers, an example of an infinite set.

DEFINITION. A set X is said to be countably infinite if there exists a bijective mapping from X to $\mathbb{N}$. A set X is said to be countable if it is finite or countably infinite.

REMARK. Suppose that X is countably infinite. Then we can write

$$X = \{x_1, x_2, x_3, \ldots\}.$$

Here we understand that there is a bijective mapping $\phi : X \to \mathbb{N}$ where $\phi(x_n) = n$ for every $n \in \mathbb{N}$.

THEOREM 1G. *A countable union of countable sets is countable.*

PROOF. Let I be a countable index set, where for each $i \in I$, the set X_i is countable. Either (a) I is finite; or (b) I is countably infinite. We shall only consider (b), since (a) needs only minor modification. Since I is countably infinite, there exists a bijective mapping from I to $\mathbb{N}$. We may therefore assume, without loss of generality, that $I = \mathbb{N}$. For each $n \in \mathbb{N}$, since X_n is countable, we may write

$$X_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\},$$

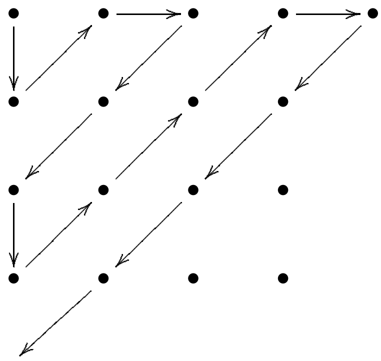
with the convention that if X_n is finite, then the sequence $a_{n1}, a_{n2}, a_{n3}, \ldots$ is constant from some point onwards. Hence we have a doubly infinite array

$$\begin{matrix} a_{11} & a_{12} & a_{13} & \cdots \\ a_{21} & a_{22} & a_{23} & \cdots \\ a_{31} & a_{32} & a_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

of elements of the set

$$X = \bigcup_{n \in \mathbb{N}} X_n.$$

We now list these elements in the order indicated by



but discarding duplicates. If X is infinite, the above clearly gives rise to a bijection from X to $\mathbb{N}$. ◯

EXAMPLE 1.4.1. The set $\mathbb{Z}$ is countable. Simply note that $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-1, -2, -3, \ldots\}$.

EXAMPLE 1.4.2. The set $\mathbb{Q}$ is countable. To see this, note that any $x \in \mathbb{Q}$ can be written in the form p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. It is easy to see that for every $n \in \mathbb{N}$, the set $Q_n = \{p/n : p \in \mathbb{Z}\}$ is countable. The result follows from Theorem 1G on observing that

$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} Q_n.$$

Suppose that two sets X_1 and X_2 are both countably infinite. Since both can be mapped to $\mathbb{N}$ bijectively, it follows that each can be mapped to the other bijectively. In this case, we say that the two sets X_1 and X_2 have the same cardinality. Cardinality can be considered as a way of measuring size. If there exists a one-to-one mapping from X_1 to X_2 and no one-to-one mapping from X_2 to X_1, then we say that X_2 has greater cardinality than X_1. For example, $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality. We shall now show that $\mathbb{R}$ has greater cardinality than $\mathbb{Q}$.

To do so, we first need an intermediate result.

THEOREM 1H. *Any subset of a countable set is countable.*

PROOF. Let X be a countable set. If X is finite, then the result is trivial. We therefore assume that X is countably infinite, so that we can write

$$X = \{x_1, x_2, x_3, \ldots\}.$$

Let Y be a subset of X. If Y is finite, then the result is trivial. If Y is infinite, then we can write

$$Y = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\},$$

where

$$n_1 = \min\{n \in \mathbb{N} : x_n \in Y\},$$

and where, for every $p \geq 2$,

$$n_p = \min\{n > n_{p-1} : x_n \in Y\}.$$

The result follows. ◯

THEOREM 1J. *The set* $\mathbb{R}$ *is not countable.*

PROOF. In view of Theorem 1H, it suffices to show that the set $[0,1)$ is not countable. Suppose on the contrary that $[0,1)$ is countable. Then we can write

$$[0,1) = \{x_1, x_2, x_3, \ldots\}. \tag{1}$$

For each $n \in \mathbb{N}$, we express x_n in decimal notation in the form

$$x_n = .x_{n1}x_{n2}x_{n3}\ldots,$$

where for each $k \in \mathbb{N}$, the digit $x_{nk} \in \{0,1,2,\ldots,9\}$. Note that this expression may not be unique, but it does not matter, as we simply choose one. We now have

$$\begin{aligned} x_1 &= .x_{11}x_{12}x_{13}\ldots, \\ x_2 &= .x_{21}x_{22}x_{23}\ldots, \\ x_3 &= .x_{31}x_{32}x_{33}\ldots, \\ &\vdots \end{aligned}$$

Let $y = .y_1y_2y_3\ldots$, where for each $n \in \mathbb{N}$, $y_n \in \{0,1,2,\ldots,9\}$ and $y_n \equiv x_{nn} + 5 \pmod{10}$. Then clearly $y \neq x_n$ for any $n \in \mathbb{N}$. But $y \in [0,1)$, contradicting (1). ◯

EXAMPLE 1.4.3. Note that the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers is not countable. It follows that in the sense of cardinality, there are far more irrational numbers than rational numbers.

1.5. Cardinal Numbers

It is easy to show that there exists a bijective mapping from a finite set X_1 to a finite set X_2 if and only if the two sets X_1 and X_2 have the same number of elements. In this case, we say that the two sets have the same cardinality. It is then convenient to denote the cardinality of a finite set by the number of elements that it contains, and take the non-negative integers to represent the finite cardinal numbers.

This may appear to be satisfactory. Strictly speaking, we need the following axiom which covers infinite sets as well.

POSTULATE OF THE CARDINAL NUMBERS. *For every set X, there exists an object $|X|$, called the cardinal number of X, which satisfies the following property: For any two sets X and Y, we have $|X| = |Y|$ if and only if there exists a bijective mapping $f : X \to Y$.*

REMARKS. (1) Note that the cardinal number of an infinite set cannot be equal to the cardinal number of a finite set, since there cannot be a bijective mapping from an infinite set to a finite set.

(2) We write $\aleph_0 = |\mathbb{N}|$ and $c = |\mathbb{R}|$.

(3) Note that $|X| = \aleph_0$ for any countably infinite set X.

(4) In view of Theorem 1J, we have $\aleph_0 \neq c$.

Definition. Suppose that X and Y are sets.
(1) We say that $|X| \leq |Y|$ if there exists an injective mapping $f : X \to Y$.
(2) We say that $|X| < |Y|$ when $|X| \leq |Y|$ and $|X| \neq |Y|$.

Remarks. (1) Note that the definition is consistent with our observation at the beginning of this section and the usual meaning of the inequalities $\leq$ and $<$ when applied to non-negative integers.

(2) Note that $|X| < |Y|$ for every finite set X and infinite set Y.

The purpose of this section is to prove the following famous result. The special case when the sets X and Y are finite is obvious.

THEOREM 1K. (CANTOR-BERNSTEIN-SCHRÖDER THEOREM) *Suppose that X and Y are sets. Suppose further that $|X| \leq |Y|$ and $|Y| \leq |X|$. Then $|X| = |Y|$.*

Proof. Since $|X| \leq |Y|$ and $|Y| \leq |X|$, there exist injective mappings $f : X \to Y$ and $g : Y \to X$. For every $x \in X$, exactly one of the following holds:

- For every $y \in Y$, we have $g(y) \neq x$. In this case, we shall say that x has no predecessor.
- There exists a unique $y_1 \in Y$ such that $g(y_1) = x$. Here the uniqueness follows from the injective property of the mapping $g : Y \to X$. In this case, we shall say that y_1 is the predecessor of x.

Similarly, for every $y \in Y$, exactly one of the following holds:

- For every $x \in X$, we have $f(x) \neq y$. In this case, we shall say that y has no predecessor.
- There exists a unique $x_1 \in X$ such that $f(x_1) = y$. Here the uniqueness follows from the injective property of the mapping $f : X \to Y$. In this case, we shall say that x_1 is the predecessor of y.

Observe also that every $x \in X$ is the predecessor of a unique element $f(x)$ in Y, and that every $y \in Y$ is the predecessor of a unique element $g(y)$ in X. It follows that for every element $x \in X$, we can construct a chain as follows:

$$\ldots \xrightarrow{f} y_2 \xrightarrow{g} x_1 \xrightarrow{f} y_1 \xrightarrow{g} x \xrightarrow{f} f(x) \xrightarrow{g} g(f(x)) \xrightarrow{f} \ldots$$

Here y_1 is the predecessor of x, x_1 is the predecessor of y_1, y_2 is the predecessor of x_1, and so on. Note that the chain does not terminate on the right, but may terminate on the left at an element with no predecessor. Similarly, for every element $y \in Y$, we can construct a chain as follows:

$$\ldots \xrightarrow{g} x_2 \xrightarrow{f} y_1 \xrightarrow{g} x_1 \xrightarrow{f} y \xrightarrow{g} g(y) \xrightarrow{f} f(g(y)) \xrightarrow{g} \ldots$$

Here x_1 is the predecessor of y, y_1 is the predecessor of x_1, x_2 is the predecessor of y_1, and so on. Again the chain does not terminate on the right, but may terminate on the left at an element with no predecessor. It is easy to see that no element of X or Y can be in two distinct chains. We now define a mapping $h : X \to Y$ as follows:

- For any element $x \in X$ whose chain does not terminate on the left or terminates on the left with an element in X with no predecessor, we let $h(x) = f(x)$.
- For any element $x \in X$ whose chain terminates on the left with an element of Y with no predecessor, we let $h(x) = y$, where $g(y) = x$, so that y is the predecessor of x.

Note that the function $h : X \to Y$ defined in this way gives a one-to-one correspondence between the elements of X and the elements of Y in each chain, and so gives a one-to-one correspondence between the elements of X and Y. ◯

Problems for Chapter 1

1. Suppose that $a, b \in \mathbb{R}$ satisfy $a > 0$ and $b < 0$. Show that $ab < 0$.

2. Suppose that $a, b \in \mathbb{R}$ satisfy $b < a < 0$. Show that $b^{-1} > a^{-1}$.

3. For each of the following sets A, determine whether $\sup A$ and $\inf A$ exist, and find their values if appropriate and determine also whether $\sup A$ and $\inf A$ belong to the set A:
   a) $A = \{n^{-1} : n \in \mathbb{N}\}$
   b) $A = \{(|n| + 1)^{-2} : n \in \mathbb{Z}\}$
   c) $A = \{n + n^{-1} : n \in \mathbb{N}\}$
   d) $A = \{2^{-m} - 3^n : m, n \in \mathbb{N}\}$
   e) $A = \{x \in \mathbb{R} : x^3 - 4x < 0\}$
   f) $A = \{1 + x^2 : x \in \mathbb{R}\}$

4. Suppose that A is a bounded set of real numbers, and that B is a non-empty subset of A. Explain why $\inf A \le \inf B \le \sup B \le \sup A$.

5. Suppose that $a, b \in \mathbb{R}$ satisfy $a < b + n^{-1}$ for every $n \in \mathbb{N}$. Prove that $a \le b$.

6. a) Suppose that $x \le a$ for every $x \in A$. Show that $\sup A \le a$.
   b) Show that the corresponding statement with $\le$ replaced by $<$ does not hold.

7. Suppose that A and B are non-empty sets of real numbers bounded above and below.
   a) Let $A \cup B = \{x : x \in A \text{ or } x \in B\}$. Prove that
   $$\sup(A \cup B) = \max\{\sup A, \sup B\} \qquad \text{and} \qquad \inf(A \cup B) = \min\{\inf A, \inf B\}.$$
   b) Discuss the case $A \cap B = \{x : x \in A \text{ and } x \in B\}$.

8. Suppose that A and B are non-empty sets of real numbers bounded above and below.
   a) Let $A + B = \{a + b : a \in A \text{ and } b \in B\}$. Prove that
   $$\sup(A + B) = \sup A + \sup B \qquad \text{and} \qquad \inf(A + B) = \inf A + \inf B.$$
   b) Discuss the case $A - B = \{a - b : a \in A \text{ and } b \in B\}$.

9. Suppose that A and B are non-empty sets of positive real numbers bounded above and below.
   a) Let $AB = \{ab : a \in A \text{ and } b \in B\}$. Prove that
   $$\sup(AB) = (\sup A)(\sup B) \qquad \text{and} \qquad \inf(AB) = (\inf A)(\inf B).$$
   b) Discuss the case when the sets A and B can contain negative real numbers.

10. Suppose that A is a non-empty set of real numbers bounded above and below. For any real number $k \in \mathbb{R}$, consider the set $kA = \{ka : a \in A\}$. What can we say about $\sup(kA)$ and $\inf(kA)$?

11. Prove that the cartesian product of two countable sets is countable.

12. A rational point in $\mathbb{C}$ is one with rational real and imaginary parts. Prove that the set of all rational points in $\mathbb{C}$ is countable.

13. Prove that any isolated point set in $\mathbb{C}$ is countable.

14. a) Find a bijection from $(0, 1)$ to $(0, \infty)$.
    b) Find a bijection from $(-1, 1)$ to $\mathbb{R}$.
    c) Suppose that $A, B \in \mathbb{R}$ with $A < B$. Find a bijection from (A, B) to $(-1, 1)$.
    d) What is the cardinality of the interval (A, B) in part (c)?

15. A real algebraic number is any real solution of a polynomial equation with coefficients in $\mathbb{Z}$. Prove that the set of all real algebraic numbers is countable.