FUNDAMENTALS OF ANALYSIS

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Chapter 2

SEQUENCES AND LIMITS

2.1. Introduction

A sequence is a set of terms occurring in order. In simple cases, a sequence is defined by an explicit formula giving the n-th term z_n in terms of n. We shall simply refer to the sequence z_n. For example, $z_n = 1/n$ represents the sequence

$$1, \tfrac{1}{2}, \tfrac{1}{3}, \tfrac{1}{4}, \ldots.$$

We shall only be concerned with the case when all the terms of a sequence are real or complex numbers, so that throughout this chapter, z_n represents a real or complex sequence. We often simply refer to a sequence z_n.

It is not necessary to start the sequence with z_1. However, the set $\mathbb{N}$ of all natural numbers is a convenient tool to indicate the order in which the terms of the sequence occur.

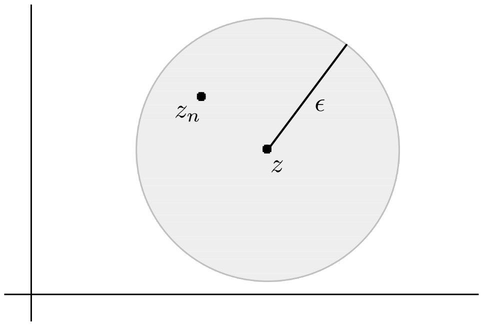
Remark. Formally, a complex sequence is a function of the form $f : \mathbb{N} \to \mathbb{C}$, where for every $n \in \mathbb{N}$, we write $f(n) = z_n$.

Definition. We say that a sequence z_n converges to a finite limit $z \in \mathbb{C}$, denoted by $z_n \to z$ as $n \to \infty$ or by

$$\lim_{n\to\infty} z_n = z,$$

if, given any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{R}$, depending on ϵ, such that $|z_n - z| < \epsilon$ whenever $n > N$. Furthermore, we say that a sequence z_n is convergent if it converges to some finite limit z as $n \to \infty$, and that a sequence z_n is divergent if it is not convergent.

Remark. The quantity $|z_n - z|$ measures the difference between z_n and its intended limit z. The definition thus says that this difference can be made as small as we like, provided that n is large enough. It follows that the convergence is not affected by the initial terms. Observe that the inequality $|z_n - z| < \epsilon$ is equivalent to saying that the point z_n lies inside a circle of radius ϵ and centred at z.



In the case when $z_n = x_n$ and $z = x$ are real, the inequality $|x_n - x| < \epsilon$ is equivalent to the inequalities $x - \epsilon < x_n < x + \epsilon$, so that x_n lies in the open interval $(x - \epsilon, x + \epsilon)$.

Example 2.1.1. Consider the sequence $z_n = 1/n$. Then $z_n \to 0$ as $n \to \infty$. We have

$$|z_n - 0| = \left|\frac{1}{n} - 0\right| = \frac{1}{n} < \epsilon$$

whenever $n > 1/\epsilon$. We may take $N = 1/\epsilon$.

Example 2.1.2. Consider the sequence $z_n = \mathrm{i}^n/n^2$. Then $z_n \to 0$ as $n \to \infty$. We have

$$|z_n - 0| = \left|\frac{\mathrm{i}^n}{n^2} - 0\right| = \frac{1}{n^2} < \epsilon$$

whenever $n > \sqrt{1/\epsilon}$. We may take $N = \sqrt{1/\epsilon}$.

Example 2.1.3. Consider the sequence $z_n = (n + 2\mathrm{i})/n$. Then $z_n \to 1$ as $n \to \infty$. We have

$$|z_n - 1| = \left|\frac{n + 2\mathrm{i}}{n} - 1\right| = \left|\frac{2\mathrm{i}}{n}\right| = \frac{2}{n} < \epsilon$$

whenever $n > 2/\epsilon$. We may take $N = 2/\epsilon$.

Example 2.1.4. Consider the sequence $z_n = \sqrt{(n + 1)/n}$. Then $z_n \to 1$ as $n \to \infty$. We have

$$|z_n - 1| = \left|\sqrt{\frac{n + 1}{n}} - 1\right| = \frac{\frac{n+1}{n} - 1}{\sqrt{\frac{n+1}{n}} + 1} < \frac{1}{2n} < \epsilon$$

whenever $n > 1/2\epsilon$. We may take $N = 1/2\epsilon$.

Example 2.1.5. Consider the sequence $z_n = (2n + 3)/(3n + 4)$. Then $z_n \to 2/3$ as $n \to \infty$. We have

$$\left|z_n - \frac{2}{3}\right| = \left|\frac{2n + 3}{3n + 4} - \frac{2}{3}\right| = \frac{1}{3(3n + 4)} < \frac{1}{9n} < \epsilon$$

whenever $n > 1/9\epsilon$. We may take $N = 1/9\epsilon$.

A simple and immediate consequence of our definition of convergence is the following result.

THEOREM 2A. *The limit of a convergent sequence is unique.*

Proof. Suppose that $z_n \to z'$ and $z_n \to z''$ as $n \to \infty$. Then given any $\epsilon > 0$, there exist $N', N'' \in \mathbb{R}$ such that

$$|z_n - z'| < \epsilon \qquad \text{whenever } n > N',$$

and

$$|z_n - z''| < \epsilon \qquad \text{whenever } n > N''.$$

Let $N = \max\{N', N''\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|z' - z''| = |(z' - z_n) + (z_n - z'')| \leq |z_n - z'| + |z_n - z''| < 2\epsilon.$$

Now $|z' - z''|$ is a non-negative constant less than any $2\epsilon > 0$, so we must have $|z' - z''| = 0$, whence $z' = z''$. ◯

Definition. A sequence z_n is said to be bounded if there exists a number $M \in \mathbb{R}$ such that $|z_n| \leq M$ for every $n \in \mathbb{N}$.

Example 2.1.6. The sequence $z_n = 1/n$ is bounded, with $|z_n| \leq 1$ for every $n \in \mathbb{N}$.

Example 2.1.7. The sequence $z_n = \mathrm{i}^n/n^2$ is bounded, with $|z_n| \leq 1$ for every $n \in \mathbb{N}$.

Example 2.1.8. The sequence $z_n = (n + 2\mathrm{i})/n$ is bounded, with $|z_n| \leq \sqrt{5}$ for every $n \in \mathbb{N}$.

Example 2.1.9. The sequence $z_n = \sqrt{(n+1)/n}$ is bounded, with $|z_n| \leq \sqrt{2}$ for every $n \in \mathbb{N}$.

Example 2.1.10. The sequence $z_n = (2n + 3)/(3n + 4)$ is bounded, with $|z_n| \leq 5/3$ for every $n \in \mathbb{N}$.

Note that the bounded sequences in Examples 2.1.6–2.1.10 are precisely the convergent sequences in Examples 2.1.1–2.1.5 respectively. They illustrate the fact that convergence implies boundedness. More precisely, we have the following result.

THEOREM 2B. *A convergent sequence is bounded.*

Proof. Suppose that $z_n \to z$ as $n \to \infty$. Then there exists $N \in \mathbb{N}$ such that $|z_n - z| < 1$ for every $n > N$. Hence

$$|z_n| < |z| + 1 \qquad \text{whenever } n > N.$$

Let $M = \max\{|z_1|, \ldots, |z_N|, |z| + 1\}$. Then clearly $|z_n| \leq M$ for every $n \in \mathbb{N}$. ◯

The next example shows that a bounded sequence is not necessarily convergent.

Example 2.1.11. The sequence $z_n = (-1)^n$ is bounded, with $|z_n| \leq 1$ for every $n \in \mathbb{N}$. We now show that this sequence is not convergent. Let z be any given complex number. We shall show that the sequence z_n does not converge to z. Note first of all that for every $n \in \mathbb{N}$, we have $|z_{n+1} - z_n| = 2$. It follows that

$$2 = |z_{n+1} - z_n| = |(z_{n+1} - z) + (z - z_n)| \leq |z_{n+1} - z| + |z_n - z|.$$

This means that for every $n \in \mathbb{N}$, at least one of the two inequalities $|z_{n+1} - z| \geq 1$ and $|z_n - z| \geq 1$ must hold. Hence the condition for convergence cannot be satisfied with $\epsilon = 1$.

The next result shows that we can do arithmetic on limits.

THEOREM 2C. *Suppose that $z_n \to z$ and $w_n \to w$ as $n \to \infty$. Then*
(a) $z_n + w_n \to z + w$ as $n \to \infty$;
(b) $z_n w_n \to zw$ as $n \to \infty$; and
(c) if $w \neq 0$, then $z_n/w_n \to z/w$ as $n \to \infty$.

REMARK. Let $w_n = 1/n$ and $t_n = (-1)^n$. Then $w_n \to 0$ as $n \to \infty$, but t_n does not converge as $n \to \infty$. On the other hand, it is easy to check that $z_n = w_n t_n \to 0$ as $n \to \infty$. Note now that $t_n = z_n/w_n$, but since $w_n \to 0$ as $n \to \infty$, we cannot use Theorem 2C(c).

PROOF OF THEOREM 2C. (a) We shall use the inequality

$$|(z_n + w_n) - (z + w)| \leq |z_n - z| + |w_n - w|.$$

Given any $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{R}$ such that

$$|z_n - z| < \epsilon/2 \qquad \text{whenever } n > N_1,$$

and

$$|w_n - w| < \epsilon/2 \qquad \text{whenever } n > N_2.$$

Let $N = \max\{N_1, N_2\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|(z_n + w_n) - (z + w)| \leq |z_n - z| + |w_n - w| < \epsilon.$$

(b) We shall use the inequality

$$\begin{aligned} |z_n w_n - zw| &= |z_n w_n - z_n w + z_n w - zw| \\ &= |z_n(w_n - w) + (z_n - z)w| \\ &\leq |z_n||w_n - w| + |w||z_n - z|. \end{aligned}$$

Since $z_n \to z$ as $n \to \infty$, there exists $N_1 \in \mathbb{R}$ such that

$$|z_n - z| < 1 \qquad \text{whenever } n > N_1,$$

so that

$$|z_n| < |z| + 1 \qquad \text{whenever } n > N_1.$$

On the other hand, given any $\epsilon > 0$, there exist $N_2, N_3 \in \mathbb{R}$ such that

$$|z_n - z| < \frac{\epsilon}{2(|w| + 1)} \qquad \text{whenever } n > N_2,$$

and

$$|w_n - w| < \frac{\epsilon}{2(|z| + 1)} \qquad \text{whenever } n > N_3.$$

Let $N = \max\{N_1, N_2, N_3\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|z_n w_n - zw| \leq |z_n||w_n - w| + |w||z_n - z| < \epsilon.$$

(c) We shall first show that $1/w_n \to 1/w$ as $n \to \infty$. To do this, we shall use the identity

$$\left|\frac{1}{w_n} - \frac{1}{w}\right| = \frac{|w_n - w|}{|w_n||w|}.$$

Since $w \neq 0$ and $w_n \to w$ as $n \to \infty$, there exists $N_1 \in \mathbb{R}$ such that

$$|w_n - w| < |w|/2 \qquad \text{whenever } n > N_1,$$

so that

$$|w_n| > |w|/2 \qquad \text{whenever } n > N_1.$$

On the other hand, given any $\epsilon > 0$, there exists $N_2 \in \mathbb{R}$ such that

$$|w_n - w| < |w|^2\epsilon/2 \qquad \text{whenever } n > N_2.$$

Let $N = \max\{N_1, N_2\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$\left|\frac{1}{w_n} - \frac{1}{w}\right| = \frac{|w_n - w|}{|w_n||w|} \leq \frac{2|w_n - w|}{|w|^2} < \epsilon.$$

We now apply part (b) to z_n and $1/w_n$ to get the desired result. ○

Definition. We say that a sequence z_n diverges to ∞ as $n \to \infty$, denoted by $z_n \to \infty$ as $n \to \infty$, if, for every $E > 0$, there exists $N \in \mathbb{R}$ such that $|z_n| > E$ whenever $n > N$.

Remarks. (1) It can be shown that $z_n \to \infty$ as $n \to \infty$ if and only if $1/z_n \to 0$ as $n \to \infty$.

(2) Note that Theorem 2C does not apply in the case when a sequence diverges to ∞.

Example 2.1.12. The sequences $z_n = n$, $z_n = n^2$ and $z_n = (-1)^n n$ all satisfy $z_n \to \infty$ as $n \to \infty$.

Example 2.1.13. Suppose that x_n is a sequence of positive terms such that $x_n \to 0$ as $n \to \infty$. For every fixed $m \in \mathbb{N}$, we have $x_n^m \to 0$ as $n \to \infty$, in view of Theorem 2C(b). For every negative integer m, we have $x_n^m \to \infty$ as $n \to \infty$, noting that $x_n > 0$ for every $n \in \mathbb{N}$. How about $m = 0$?

2.2. Real Sequences

Real sequences are particularly interesting since the real numbers are ordered, unlike the complex numbers. This enables us to establish special results for convergence which apply only to real sequences.

We begin with a simple example. Imagine that you have a ham sandwich, and you do the most disgusting thing of squeezing the two slices of bread together. Where does the ham go?

THEOREM 2D. (SQUEEZING PRINCIPLE) *Suppose that* $x_n \to x$ *and* $y_n \to x$ *as* $n \to \infty$. *Suppose further that* $x_n \leq a_n \leq y_n$ *for every* $n \in \mathbb{N}$. *Then* $a_n \to x$ *as* $n \to \infty$.

Example 2.2.1. Consider the sequence

$$a_n = \frac{4n+3}{4n^2+3n+1}.$$

Then

$$\frac{1}{2n} = \frac{4n}{8n^2} < \frac{4n+3}{4n^2+3n+1} < \frac{4n+3+n^{-1}}{4n^2+3n+1} = \frac{1}{n}.$$

Writing

$$x_n = \frac{1}{2n} \qquad \text{and} \qquad y_n = \frac{1}{n},$$

we have that $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$. Hence $a_n \to 0$ as $n \to \infty$.

EXAMPLE 2.2.2. Consider the sequence $a_n = n^{-1}\cos n$. If $x_n = -1/n$ and $y_n = 1/n$, then clearly $x_n \le a_n \le y_n$ for every $n \in \mathbb{N}$. Since $x_n \to 0$ and $y_n \to 0$ as $n \to \infty$, we have $a_n \to 0$ as $n \to \infty$.

EXAMPLE 2.2.3. It is important that x_n and y_n converge to the same limit. For example, if $x_n = -1$ and $y_n = 1$ for every $n \in \mathbb{N}$, then both x_n and y_n converge as $n \to \infty$. Let $a_n = (-1)^n$. Then $x_n \le a_n \le y_n$ for every $n \in \mathbb{N}$. Note from Example 2.1.11 that a_n does not converge as $n \to \infty$. In this case, the hypotheses of Theorem 2D are not satisfied. Note that x_n and y_n converge to different limits, so no squeezing occurs.

EXAMPLE 2.2.4. Consider the sequence $x_n = a^n$, where $a \in \mathbb{R}$. There are various cases:

- If $a = 1$, then $x_n = 1$ for every $n \in \mathbb{N}$, so that $x_n \to 1$ as $n \to \infty$.
- If $a = 0$, then $x_n = 0$ for every $n \in \mathbb{N}$, so that $x_n \to 0$ as $n \to \infty$.
- If $a > 1$, then $a = 1 + k$, where $k > 0$. Then

$$|a^n| = (1+k)^n \ge 1 + kn > E \qquad \text{for every } n > \frac{E-1}{k}.$$

  It follows that $x_n \to \infty$ as $n \to \infty$.
- If $0 < a < 1$, then $a = 1/b$, where $b > 1$. Hence $1/x_n \to \infty$ as $n \to \infty$. It follows that $x_n \to 0$ as $n \to \infty$.
- If $-1 < a < 0$, then $a = -b$, where $0 < b < 1$. We then have $b^n \to 0$ as $n \to \infty$. Also, $-b^n \le x_n \le b^n$ for every $n \in \mathbb{N}$. It follows from the Squeezing principle that $x_n \to 0$ as $n \to \infty$.
- If $a = -1$, then $x_n = (-1)^n$ does not converge as $n \to \infty$.
- If $a < -1$, then $a = 1/b$ where $-1 < b < 0$. Hence $1/x_n \to 0$ as $n \to \infty$. It follows that $x_n \to \infty$ as $n \to \infty$.

PROOF OF THEOREM 2D. By Theorem 2C, $y_n - x_n \to 0$ as $n \to \infty$. It follows that given any $\epsilon > 0$, there exist $N', N'' \in \mathbb{R}$ such that

$$|y_n - x_n| < \epsilon/2 \qquad \text{whenever } n > N',$$

and

$$|x_n - x| < \epsilon/2 \qquad \text{whenever } n > N''.$$

Let $N = \max\{N', N''\} \in \mathbb{R}$. It follows that whenever $n > N$, we have

$$|a_n - x| \le |a_n - x_n| + |x_n - x| \le |y_n - x_n| + |x_n - x| < \epsilon.$$

Hence $a_n \to x$ as $n \to \infty$. ◯

Our next task is to study monotonic sequences which are particularly interesting.

DEFINITION. Let x_n be a real sequence.

(1) We say that x_n is increasing if $x_{n+1} \ge x_n$ for every $n \in \mathbb{N}$.
(2) We say that x_n is decreasing if $x_{n+1} \le x_n$ for every $n \in \mathbb{N}$.
(3) We say that x_n is bounded above if there exists $B \in \mathbb{R}$ such that $x_n \le B$ for every $n \in \mathbb{N}$.
(4) We say that x_n is bounded below if there exists $b \in \mathbb{R}$ such that $x_n \ge b$ for every $n \in \mathbb{N}$.

REMARK. Note that a real sequence is bounded if and only if it is bounded above and below.

THEOREM 2E. *Suppose that x_n is an increasing real sequence.*
(a) If x_n is bounded above, then x_n converges as $n \to \infty$.
(b) If x_n is not bounded above, then $x_n \to \infty$ as $n \to \infty$.

THEOREM 2F. *Suppose that x_n is a decreasing real sequence.*
(a) If x_n is bounded below, then x_n converges as $n \to \infty$.
(b) If x_n is not bounded below, then $x_n \to \infty$ as $n \to \infty$.

Proof of Theorem 2E. (a) Suppose that the sequence x_n is bounded above. Then the set

$$S = \{x_n : n \in \mathbb{N}\}$$

is a non-empty set of real numbers which is bounded above. Let $x = \sup S$. We shall show that $x_n \to x$ as $n \to \infty$. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_N > x - \epsilon$. Since the sequence x_n is increasing and bounded above by x, it follows that whenever $n > N$, we have $x \geq x_n \geq x_N > x - \epsilon$, so that $|x_n - x| < \epsilon$.

(b) Suppose that the sequence x_n is not bounded above. Then for every $E > 0$, there exists $N \in \mathbb{N}$ such that $x_N > E$. Since the sequence x_n is increasing, it follows that $|x_n| = x_n \geq x_N > E$ for every $n > N$. Hence $x_n \to \infty$ as $n \to \infty$. ○

Example 2.2.5. The sequence $x_n = 3 - 1/n$ is increasing and bounded above. It is not too difficult that the smallest real number $B \in \mathbb{R}$ such that $x_n \leq B$ for every $n \in \mathbb{N}$ is 3. It is easy to show that $x_n \to 3$ as $n \to \infty$.

Example 2.2.6. Consider the sequence $x_n = 1 + a + a^2 + \ldots + a^n$. Then $x_n = n + 1$ if $a = 1$ and

$$x_n = \frac{1 - a^{n+1}}{1 - a} \qquad \text{if } a \neq 1.$$

Suppose that $a > 0$. Then x_n is increasing. If $0 < a < 1$, then $x_n < 1/(1 - a)$ for all $n \in \mathbb{N}$, and so x_n converges as $n \to \infty$. If $a \geq 1$, then x_n is not bounded above, so that $x_n \to \infty$ as $n \to \infty$. In fact, if $a \neq 1$, then the convergence or divergence of x_n depends on the convergence and divergence of a^{n+1}, which we have considered before in Example 2.2.4.

Example 2.2.7. Consider the sequence

$$x_n = 1 + \frac{1}{1!} + \frac{1}{2!} + \ldots + \frac{1}{n!}.$$

Clearly x_n is an increasing sequence. On the other hand,

$$\begin{aligned} x_n &= 1 + 1 + \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \ldots + \frac{1}{(n-1)n} \\ &= 1 + 1 + \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \ldots + \left(\frac{1}{n-1} - \frac{1}{n}\right) \\ &= 3 - \frac{1}{n} < 3, \end{aligned}$$

so that x_n is bounded above. Unfortunately, it is very hard to find the smallest real number $B \in \mathbb{R}$ such that $x_n \leq B$ for every $n \in \mathbb{N}$. While Theorem 2E tells us that the sequence x_n converges, it does not tell us the precise value of the limit. In fact, the limit in this case is the number e.

2.3. Tests for Convergence

We first of all apply our knowledge of real sequences in Section 2.2 to study complex sequences.

THEOREM 2G. *Suppose that x_n and y_n are real sequences and $z_n = x_n + \mathrm{i}y_n$. Then*

$$z_n \to z = x + \mathrm{i}y \qquad \text{as } n \to \infty$$

if and only if

$$x_n \to x \quad \text{and} \quad y_n \to y \qquad \text{as } n \to \infty.$$

Proof. (⇒) Suppose first of all that $z_n \to z = x + \mathrm{i}y$ as $n \to \infty$. Then given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$|z_n - z| < \epsilon \qquad \text{whenever } n > N.$$

Observe now that

$$|x_n - x| = \sqrt{(x_n - x)^2} \le \sqrt{(x_n - x)^2 + (y_n - y)^2} = |z_n - z|.$$

It follows that

$$|x_n - x| < \epsilon \qquad \text{whenever } n > N.$$

Similarly,

$$|y_n - y| < \epsilon \qquad \text{whenever } n > N.$$

(⇐) Suppose next that $x_n \to x$ and $y_n \to y$ as $n \to \infty$. Then given any $\epsilon > 0$, there exist $N_1, N_2 \in \mathbb{R}$ such that

$$|x_n - x| < \epsilon/2 \qquad \text{whenever } n > N_1,$$

and

$$|y_n - y| < \epsilon/2 \qquad \text{whenever } n > N_2.$$

Observe now that

$$|z_n - z| = |(x_n + \mathrm{i}y_n) - (x + \mathrm{i}y| \le |x_n - x| + |y_n - y|.$$

Let $N = \max\{N_1, N_2\} \in \mathbb{R}$. It follows that

$$|z_n - z| < \epsilon \qquad \text{whenever } n > N.$$

This completes the proof. ○

We now return to Theorem 2D. It turns out often that the sequences x_n and y_n in Theorem 2D can be constructed artificially. An example is the following result.

THEOREM 2H. (RATIO TEST) *Suppose that the sequence z_n satisfies*

$$\left|\frac{z_{n+1}}{z_n}\right| \to \ell \qquad \text{as } n \to \infty. \tag{1}$$

(a) If $\ell < 1$, then $z_n \to 0$ as $n \to \infty$.
(b) If $\ell > 1$, then $z_n \to \infty$ as $n \to \infty$.

Proof. (a) Suppose that $\ell < 1$. Write $L = \frac{1}{2}(1 + \ell)$. Then clearly $\ell < L < 1$. On the other hand, it follows from (1) and taking $\epsilon = \frac{1}{2}(1 - \ell) > 0$ that there exists an integer N_0 such that

$$\left|\left|\frac{z_{n+1}}{z_n}\right| - \ell\right| < \frac{1 - \ell}{2} \qquad \text{whenever } n > N_0.$$

In particular, we have

$$\left|\frac{z_{n+1}}{z_n}\right| < \ell + \frac{1 - \ell}{2} = L \qquad \text{whenever } n > N_0.$$

It follows that for every $n > N_0$, we have

$$|z_n| < L|z_{n-1}| < L^2|z_{n-2}| < \ldots < L^{n-N_0}|z_{N_0}| = L^{-N_0}|z_{N_0}|L^n.$$

Let

$$M = \max_{1 \le n \le N_0} \frac{|z_n|}{L^n}.$$

Then for every $n \in \mathbb{N}$, we have

$$0 \le |z_n| \le ML^n.$$

Clearly the sequence $ML^n \to 0$ as $n \to \infty$. It follows from Theorem 2D that $|z_n| \to 0$ as $n \to \infty$, so that $z_n \to 0$ as $n \to \infty$.

(b) Suppose that $\ell > 1$. Let $w_n = 1/z_n$. Then $|w_{n+1}/w_n| \to 1/\ell$ as $n \to \infty$. It follows from (a) that $w_n \to 0$ as $n \to \infty$, so that $z_n \to \infty$ as $n \to \infty$. ◯

Remark. No firm conclusion can be drawn when $\ell = 1$, as can be seen from the following sequences which all have $\ell = 1$:
- The sequence $z_n = c$ converges to c as $n \to \infty$.
- The sequence $z_n = (-1)^n$ diverges as $n \to \infty$.
- The sequence $z_n = 1/n$ converges to 0 as $n \to \infty$.
- The sequence $z_n = n$ diverges to infinity as $n \to \infty$.
- The sequence $z_n = \mathrm{i}^n n$ diverges to infinity as $n \to \infty$.

Example 2.3.1. Consider the sequence $z_n = \dfrac{(n!)^2}{(2n)!}$. We have

$$\left|\frac{z_{n+1}}{z_n}\right| = \frac{z_{n+1}}{z_n} = \frac{((n+1)!)^2}{(2(n+1))!} \bigg/ \frac{(n!)^2}{(2n)!} = \frac{(n+1)^2}{(2n+2)(2n+1)} = \frac{n^2+2n+1}{4n^2+6n+2} \to \frac{1}{4} \qquad \text{as } n \to \infty.$$

It follows from Theorem 2H that $z_n \to 0$ as $n \to \infty$.

Example 2.3.2. Consider the sequence $z_n = \dfrac{(n!)^2}{(2n)!}5^n$. Then $|z_{n+1}/z_n| \to 5/4$ as $n \to \infty$. It follows from Theorem 2H that $z_n \to \infty$ as $n \to \infty$.

2.4. Recurrence Relations

In practice, it may not always be convenient to define a sequence explicitly. Sequences may often be defined by a relation connecting two or more successive terms. Here we shall not make a thorough study of such relations, but confine our discussion to two examples of real sequences.

Example 2.4.1. Suppose that $x_1 = 3$ and

$$x_{n+1} = \frac{4x_n + 2}{x_n + 3}$$

for every $n \in \mathbb{N}$. Note first of all that $0 < x_2 < x_1$. Suppose that $n > 1$ and $0 < x_n < x_{n-1}$. Then clearly $x_{n+1} > 0$. Furthermore,

$$x_{n+1} - x_n = \frac{4x_n + 2}{x_n + 3} - \frac{4x_{n-1} + 2}{x_{n-1} + 3} = \frac{10(x_n - x_{n-1})}{(x_n + 3)(x_{n-1} + 3)} < 0.$$

It follows from the Principle of induction that x_n is a decreasing sequence and bounded below by 0, so that x_n converges as $n \to \infty$. Suppose that $x_n \to x$ as $n \to \infty$. Then

$$x = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} \frac{4x_n + 2}{x_n + 3} = \frac{4x + 2}{x + 3}.$$

Hence $x = 2$. Note that the other solution $x = -1$ has to be discounted, since $x_n > 0$ for every $n \in \mathbb{N}$.

EXAMPLE 2.4.2. Let $s > 0$. Suppose that $x_1 > 0$ and that for $n > 1$, we have

$$x_n = \frac{1}{2}\left(x_{n-1} + \frac{s}{x_{n-1}}\right).$$

It is not difficult to show that $x_n > 0$ for every $n \in \mathbb{N}$. On the other hand, for $n > 1$, we have

$$x_n^2 = \frac{1}{4}\left(x_{n-1}^2 + \frac{s^2}{x_{n-1}^2} + 2s\right),$$

so that

$$x_n^2 - s = \frac{1}{4}\left(x_{n-1}^2 + \frac{s^2}{x_{n-1}^2} - 2s\right) = \frac{1}{4}\left(x_{n-1} - \frac{s}{x_{n-1}}\right)^2 \geq 0,$$

and so

$$x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{s}{x_n}\right) - x_n = \frac{1}{2}\left(\frac{s}{x_n} - x_n\right) = \frac{s - x_n^2}{2x_n} \leq 0.$$

It follows that, with the possible exception that $x_2 \leq x_1$ may not hold, the sequence x_n is decreasing and bounded below, so that x_n converges as $n \to \infty$. Suppose that $x_n \to x$ as $n \to \infty$. Then

$$x = \lim_{n\to\infty} x_n = \lim_{n\to\infty} \frac{1}{2}\left(x_{n-1} + \frac{s}{x_{n-1}}\right) = \frac{1}{2}\left(x + \frac{s}{x}\right),$$

so that $x^2 = s$. This gives a proof that s has a square root.

2.5. Subsequences

In this section, we discuss subsequences. Heuristically, a subsequence is obtained from a sequence by possibly omitting some of the terms, and keeping the remainder in the original order. We can make this more formal in the following way.

DEFINITION. Suppose that

$$z_1, z_2, z_3, \ldots, z_n, \ldots$$

is a sequence. Suppose further that $n_1 < n_2 < n_3 < \ldots < n_p < \ldots$ is an infinite sequence of natural numbers. Then the sequence

$$z_{n_1}, z_{n_2}, z_{n_3}, \ldots, z_{n_p}, \ldots$$

is called a subsequence of the original sequence.

EXAMPLE 2.5.1. The sequence $2, 4, 6, 8, \ldots$ of even natural numbers is a subsequence of the sequence $1, 2, 3, 4, \ldots$ of natural numbers.

EXAMPLE 2.5.2. The sequence $2, 3, 5, 7, \ldots$ of primes is not a subsequence of the sequence $1, 3, 5, 7, \ldots$ of odd natural numbers.

EXAMPLE 2.5.3. The sequence $1, 2, 3, 4, \ldots$ of natural numbers is a subsequence of the sequence $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, \ldots$.

We would like to obtain conditions under which convergent subsequences exist. We first investigate the special case of real sequences.

THEOREM 2J. *Every sequence of real numbers has either an increasing subsequence or a decreasing subsequence, possibly both.*

PROOF. We shall say that $n \in \mathbb{N}$ is a "peak" point if $x_n > x_m$ for every $m > n$. There are precisely two possibilities:

(i) Suppose that there are infinitely many peak points $n_1 < n_2 < n_3 < \ldots < n_p < \ldots$. Then

$$x_{n_1} > x_{n_2} > x_{n_3} > \ldots > x_{n_p} > \ldots$$

is a decreasing subsequence.

(ii) Suppose that there are no or only finitely many peak points. Let $n_1 = 1$ if there are no peak points, and let $n_1 = N + 1$ if N represents the largest peak point. Then n_1 is not a peak point, and so there exists $n_2 > n_1$ such that $x_{n_1} \leq x_{n_2}$. On the other hand, n_2 is not a peak point, and so there exists $n_3 > n_2$ such that $x_{n_2} \leq x_{n_3}$. Continuing inductively, we conclude that there exists an infinite sequence $n_1 < n_2 < n_3 < \ldots < n_p < \ldots$ of natural numbers such that

$$x_{n_1} \leq x_{n_2} \leq x_{n_3} \leq \ldots \leq x_{n_p} \leq \ldots$$

is an increasing subsequence. ◯

THEOREM 2K. *Every bounded sequence of real numbers has a convergent subsequence.*

PROOF. By Theorem 2J, there is either an increasing subsequence which is necessarily bounded above, or a decreasing subsequence which is necessarily bounded below. It follows from Theorem 2E and 2F that the subsequence must be convergent. ◯

EXAMPLE 2.5.4. For the sequence $x_n = (-1)^n$, it is easy to check that all increasing or decreasing subsequences of x_n are eventually constant and so convergent.

EXAMPLE 2.5.5. For the sequence $x_n = (1 + (-1)^n)n$, it is easy to check that there is an increasing subsequence $4, 8, 12, \ldots$ $(n = 2, 4, 6, \ldots)$, as well as a decreasing subsequence $0, 0, 0, \ldots$ $(n = 1, 3, 5, \ldots)$.

EXAMPLE 2.5.6. The sequence $x_n = (-1)^n n^{-1}$ is convergent with limit 0. It is easy to check that there is an increasing subsequence (n odd), as well as a decreasing subsequence (n even), and both converge to 0. Can you convince yourself that every other subsequence of x_n converges to 0 also? If not, see Theorem 2L below.

EXAMPLE 2.5.7. The sequence $x_n = n$ diverges to infinity. Can you convince yourself that every subsequence of x_n is increasing and diverges to infinity also?

We now no longer restrict our study to real sequences, and consider subsequences of sequences of complex numbers.

THEOREM 2L. *Suppose that a sequence $z_n \to z$ as $n \to \infty$. Then for every subsequence z_{n_p} of z_n, we have $z_{n_p} \to z$ as $p \to \infty$. In other words, every subsequence of a convergent sequence converges to the same limit.*

Proof. Given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$|z_n - z| < \epsilon \qquad \text{whenever } n > N.$$

Note next that $n_p \geq p$ for every $p \in \mathbb{N}$, so that $n_p > N$ whenever $p > N$. It follows that

$$|z_{n_p} - z| < \epsilon \qquad \text{whenever } p > N.$$

Hence $z_{n_p} \to z$ as $p \to \infty$. ◯

We now extend Theorem 2K to complex sequences.

THEOREM 2M. (BOLZANO-WEIERSTRASS THEOREM) *Every bounded sequence of complex numbers has a convergent subsequence.*

Proof. Suppose that z_n is a bounded sequence of complex numbers. Let x_n and y_n be real sequences such that $z_n = x_n + \mathrm{i}y_n$. Since z_n is bounded, there exists $M \in \mathbb{R}$ such that $|z_n| \leq M$ for every $n \in \mathbb{N}$. Then clearly $|x_n| \leq M$ and $|y_n| \leq M$ for every $n \in \mathbb{N}$, so that x_n and y_n are both bounded. By Theorem 2K, the sequence x_n has a convergent subsequence x_{n_p}. Consider the corresponding subsequence y_{n_p} of the sequence y_n. Clearly $|y_{n_p}| \leq M$ for every $p \in \mathbb{N}$, so that y_{n_p} is bounded. By Theorem 2K again, the sequence y_{n_p} has a convergent subsequence $y_{n_{p_s}}$. The corresponding subsequence $x_{n_{p_s}}$ of the sequence x_{n_p}, being a subsequence of a convergent sequence, is again convergent, in view of Theorem 2L. It now follows from Theorem 2G that the subsequence $z_{n_{p_s}} = x_{n_{p_s}} + \mathrm{i}y_{n_{p_s}}$ of the sequence z_n is convergent. ◯

Definition. A complex number $\zeta \in \mathbb{C}$ is said to be a limit point of a sequence z_n if there exists a subsequence z_{n_p} of z_n such that $z_{n_p} \to \zeta$ as $p \to \infty$.

Example 2.5.8. The sequence $z_n = n$ has no limit points. To see this, note that $z_n \to \infty$ as $n \to \infty$. Let $w_n = 1/z_n$. Then $w_n \to 0$ as $n \to \infty$. It follows from Theorem 2L that every subsequence of w_n converges to 0. Hence every subsequence of z_n diverges to infinity.

Example 2.5.9. The sequence $z_n = \mathrm{i}^n$ has four limit points, namely ± 1 and $\pm\mathrm{i}$.

Example 2.5.10. The sequence

$$1, \tfrac{1}{2}, \tfrac{2}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{3}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{4}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \tfrac{5}{5}, \ldots$$

has infinitely many limit points. In fact, the set of all limit points is the closed interval $[0, 1]$. This is a famous result in diophantine approximation.

Remark. Note that Theorem 2L says that a convergent sequence has exactly one limit point. Note also that the sequence $1, 2, 1, 3, 1, 4, 1, 5, \ldots$ has exactly one limit point but does not converge.

We now characterize convergence of sequences in terms of boundedness and limited points.

THEOREM 2N. *A sequence of complex numbers is convergent if and only if it is bounded and has exactly one limit point.*

Proof. ($\Rightarrow$) This is a combination of Theorems 2B and 2L.

($\Leftarrow$) Suppose that z_n is bounded and has exactly one limit point ζ. We shall show that $z_n \to \zeta$ as $n \to \infty$. Suppose on the contrary that z_n does not converge to ζ as $n \to \infty$. Then there exists a constant $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there exists $n > N$ such that $|z_n - \zeta| \geq \epsilon_0$. Putting $N = 1$, there exists $n_1 > 1$ such that $|z_{n_1} - \zeta| \geq \epsilon_0$. Putting $N = n_1$, there exists $n_2 > n_1$ such that $|z_{n_2} - \zeta| \geq \epsilon_0$. Putting $N = n_2$, there exists $n_3 > n_2$ such that $|z_{n_3} - \zeta| \geq \epsilon_0$. Proceeding inductively, we obtain a sequence $n_1 < n_2 < n_3 < \ldots < n_p < \ldots$ of natural numbers such that $|z_{n_p} - \zeta| \geq \epsilon_0$ for every $p \in \mathbb{N}$. Since

z_n is bounded, the subsequence z_{n_p} is also bounded. It follows from the Bolzano-Weierstrass theorem that z_{n_p} has a convergent subsequence $z_{n_{p_s}}$. Suppose that $z_{n_{p_s}} \to z$ as $s \to \infty$. Then clearly $z \neq \zeta$, for $|z_{n_{p_s}} - \zeta| \geq \epsilon_0$ for every $s \in \mathbb{N}$. This means that z is another limit point of the sequence z_n, contradicting the assumption that z_n has exactly one limit point. ○

Recall that the set $\mathbb{R}$ is complete, in terms of the Axiom of bound. We now study completeness from a different viewpoint.

Definition. A sequence z_n of complex numbers is said to be a Cauchy sequence if, given any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{R}$, depending on ϵ, such that $|z_m - z_n| < \epsilon$ whenever $m > n \geq N$.

It is easy to establish the following.

THEOREM 2P. *Suppose that a sequence z_n is convergent. Then z_n is a Cauchy sequence.*

Proof. Suppose that $z_n \to z$ as $n \to \infty$. Then given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$|z_n - z| < \epsilon/2 \qquad \text{whenever } n > N.$$

It follows that

$$|z_m - z_n| = |(z_m - z) + (z - z_n)| \leq |z_m - z| + |z_n - z| < \epsilon \qquad \text{whenever } m > n \geq N + 1.$$

Hence z_n is a Cauchy sequence. ○

An alternative way of saying that $\mathbb{R}$ and $\mathbb{C}$ are complete is the following result.

THEOREM 2Q. *Suppose that z_n is a Cauchy sequence. Then z_n is convergent.*

Proof. Since z_n is a Cauchy sequence, there exists $N \in \mathbb{N}$ such that

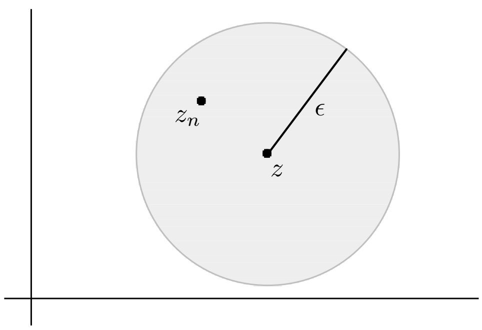
$$|z_n - z_N| < 1 \qquad \text{whenever } n \geq N,$$

so that

$$|z_n| < 1 + |z_N| \qquad \text{whenever } n \geq N.$$

Let $M = 1 + \max\{|z_1|, \ldots, |z_N|\}$. Then $|z_n| \leq M$ for every $n \in \mathbb{N}$, so that z_n is bounded. It follows from the Bolzano-Weierstrass theorem that z_n has a convergent subsequence z_{n_p}. Suppose that $z_{n_p} \to \zeta$ as $p \to \infty$. In view of Theorem 2N, it remains to show that ζ is the only limit point of z_n. Suppose on the contrary that z is another limit point of z_n. Then there exists another subsequence $z_{n'_r}$ of z_n such that $z_{n'_r} \to z$ as $r \to \infty$.

Let $\epsilon = \frac{1}{3}|\zeta - z| > 0$.



Then there exist $P, R \in \mathbb{R}$ such that

$$|z_{n_p} - \zeta| < \epsilon \qquad \text{whenever } p > P,$$

and

$$|z_{n'_r} - z| < \epsilon \qquad \text{whenever } r > R.$$

It follows that for every $p > P$ and $r > R$, we have

$$|z_{n_p} - z_{n'_r}| = |(z_{n_p} - \zeta) - (z_{n'_r} - z) + (\zeta - z)| \geq |\zeta - z| - |z_{n_p} - \zeta| - |z_{n'_r} - z| > \tfrac{1}{3}|\zeta - z|,$$

contradicting that z_n is a Cauchy sequence. ○

Problems for Chapter 2

1. Consider the sequence $z_n = \dfrac{4n+3}{5n+2}$.
   a) Make a guess for the limit of z_n as $n \to \infty$.
   b) Use the ϵ-N definition to verify that your guess is correct.

2. Show that the sequence

$$z_n = \frac{n}{2n+1} + \frac{\cos(\mathrm{e}^{\sin(25\pi n^5)} \log(n^2))}{n^3}$$

   is convergent as $n \to \infty$, find its limit and explain every step of your argument.

3. Suppose that $z_n \to \ell$ as $n \to \infty$, and that $w_n = \dfrac{z_1 + z_2 + \ldots + z_n}{n}$. Show that $w_n \to \ell$ as $n \to \infty$.
   [Hint: Consider first the case $\ell = 0$.]

4. Prove that the following sequences converge as $n \to \infty$ and find their limits except for part (d):
   a) $z_n = (n+1)^{1/4} - n^{1/4}$
   b) $z_n = \dfrac{1+2+\ldots+n}{n^2}$
   c) $z_n = \dfrac{n}{2^n}$
   d) $z_n = \dfrac{1}{n+1} + \dfrac{1}{n+2} + \ldots + \dfrac{1}{2n}$

5. Show that the real sequence $x_n = \left(1 + \dfrac{1}{n}\right)^n$ is increasing and bounded above.
   [Remark: Hence it converges. The limit is the number e.]

6. Suppose that z is a fixed complex number. Discuss the convergence and divergence of the sequence

$$z_n = \frac{z + z^n}{1 + z^n},$$

   explain every step of your argument, and take care to distinguish the four cases
   a) $|z| > 1$; b) $|z| < 1$; c) $z = 1$; d) $|z| = 1$, but $z \neq 1$.

7. A real sequence x_n is defined inductively by $x_1 = 1$ and $x_{n+1} = \sqrt{x_n + 6}$ for every $n \in \mathbb{N}$.
   a) Prove by induction that x_n is increasing, and $x_n < 3$ for every $n \in \mathbb{N}$.
   b) Deduce that x_n converges as $n \to \infty$ and find its limit.

8. Suppose that $x_1 < x_2$ and $x_{n+2} = \frac{1}{2}(x_{n+1} + x_n)$ for every $n \in \mathbb{N}$. Show that
   a) $x_{n+2} > x_n$ for every odd $n \in \mathbb{N}$;
   b) $x_{n+2} < x_n$ for every even $n \in \mathbb{N}$; and
   c) $x_n \to \frac{1}{3}(x_1 + 2x_2)$ as $n \to \infty$.

9. Find the limit points of each of the following complex sequences:
   a) $z_n = (-1)^n$
   b) $z_n = (2\mathrm{i})^n$
   c) $z_n = \left(\dfrac{1+\mathrm{i}}{\sqrt{2}}\right)^n$

10. Show that a complex sequence z_n has exactly one of the following two properties:
   a) $z_n \to \infty$ as $n \to \infty$.
   b) z_n has a convergent subsequence.
   [Hint: Assume that (a) fails. Show that (b) must then hold.]

11. Suppose that $0 < b < 1$ and that the sequence a_n satisfies the condition that $|a_{n+1} - a_n| \leq b^n$ for every $n \in \mathbb{N}$. Use Theorem 2Q to prove that a_n is convergent as $n \to \infty$.