FUNDAMENTALS OF ANALYSIS

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Chapter 3

SERIES

3.1. Introduction

Suppose that z_n is a real or complex sequence. For every $N \in \mathbb{N}$, let

$$s_N = \sum_{n=1}^{N} z_n = z_1 + \ldots + z_N.$$

We shall call

$$\sum_{n=1}^{\infty} z_n \tag{1}$$

a series, and s_N the N-th partial sum of the series.

Definition. If the sequence s_N converges to s as $N \to \infty$, then we say that the series (1) converges to the sum s and write

$$\sum_{n=1}^{\infty} z_n = s.$$

In this case, we sometimes simply say that the series (1) is convergent. On the other hand, if the sequence s_N diverges as $N \to \infty$, then we say that the series (1) is divergent.

Since the partial sums of a series form a sequence, we deduce immediately from Theorems 2P and 2Q the following useful result.

THEOREM 3A. (GENERAL PRINCIPLE OF CONVERGENCE FOR SERIES) *The series* (1) *is convergent if and only if, given any $\epsilon > 0$, there exists a number N_0 such that*

$$\left|\sum_{n=N+1}^{M} z_n\right| < \epsilon \qquad \textit{whenever } M > N \geq N_0.$$

REMARK. Note that Theorem 3A says that the series (1) is convergent if and only if the sequence s_N of partial sums forms a Cauchy sequence. To prove Theorem 3A, we simply observe that

$$\sum_{n=N+1}^{M} z_n = s_M - s_N.$$

Before we study the convergence of series in general, we first look at some very useful examples.

THEOREM 3B. (GEOMETRIC SERIES) *The real geometric series*

$$\sum_{n=1}^{\infty} x^{n-1} = 1 + x + x^2 + x^3 + \ldots$$

is convergent if and only if $|x| < 1$.

PROOF. It is easy to see that the sequence s_N of partial sums satisfies

$$s_N = \begin{cases} \dfrac{1-x^N}{1-x} & \text{if } x \neq 1; \\ N & \text{if } x = 1. \end{cases}$$

If $x = 1$, then the sequence s_N is clearly not bounded, and so is not convergent as $N \to \infty$. On the other hand, we note from Example 2.2.4 that $x^N \to 0$ as $N \to \infty$ if $|x| < 1$, so that the series is convergent in this case. Finally, we note from Example 2.2.4 again that x^N is divergent if $x > 1$ or $x \leq -1$, so that the series is divergent in these cases. ◯

THEOREM 3C. (HARMONIC SERIES) *The real harmonic series*

$$\sum_{n=1}^{\infty} n^{-k}$$

is convergent if $k > 1$ and is divergent if $k \leq 1$.

PROOF. Consider first the case $k = 1$. Clearly

$$s_N = \sum_{n=1}^{N} n^{-1}$$

is an increasing real sequence. To show that the series is divergent, it suffices, in view of Theorem 2E, to show that the sequence s_N is not bounded above. We shall achieve this by proving that

$$s_{2^m} \geq 1 + \tfrac{1}{2}m \qquad \text{for every } m \in \mathbb{N}. \tag{2}$$

The inequality is clearly true for $m = 1$, since $s_2 = \frac{3}{2}$. Suppose now that $s_{2^p} \geq 1 + \frac{1}{2}p$. Then

$$s_{2^{p+1}} = s_{2^p} + \left(\frac{1}{2^p+1} + \frac{1}{2^p+2} + \ldots + \frac{1}{2^{p+1}}\right) \geq s_{2^p} + \frac{2^p}{2^{p+1}} \geq 1 + \frac{1}{2}p + \frac{1}{2} = 1 + \frac{1}{2}(p+1).$$

The assertion (2) now follows from the Principle of induction.

Suppose next that $k < 1$. In this case, we have $n^{-k} \geq n^{-1}$ for every $n \in \mathbb{N}$, and so

$$s_N = \sum_{n=1}^{N} n^{-k} \geq \sum_{n=1}^{N} n^{-1}.$$

It therefore follows from the first part that the sequence s_N is not bounded above. Clearly s_N is an increasing real sequence. It follows from Theorem 2E that the series is divergent.

Suppose finally that $k > 1$. Again, the sequence

$$s_N = \sum_{n=1}^{N} n^{-k}$$

is an increasing sequence. To show that the series is convergent, it suffices, in view of Theorem 2E, to show that the sequence s_N is bounded above. Let $t \in \mathbb{N}$ satisfy $N < 2^t$. Then

$$\begin{aligned} s_N \leq s_{2^t-1} &= 1 + \frac{1}{2^k} + \frac{1}{3^k} + \ldots + \frac{1}{(2^t-1)^k} \\ &= 1 + \left(\frac{1}{2^k} + \frac{1}{3^k}\right) + \left(\frac{1}{4^k} + \ldots + \frac{1}{7^k}\right) + \left(\frac{1}{8^k} + \ldots + \frac{1}{15^k}\right) + \ldots + \left(\frac{1}{(2^{t-1})^k} + \ldots + \frac{1}{(2^t-1)^k}\right) \\ &< 1 + \frac{2}{2^k} + \frac{4}{4^k} + \frac{8}{8^k} + \ldots + \frac{2^{t-1}}{(2^{t-1})^k} \\ &= 1 + \frac{1}{2^{k-1}} + \left(\frac{1}{2^{k-1}}\right)^2 + \left(\frac{1}{2^{k-1}}\right)^3 + \ldots + \left(\frac{1}{2^{k-1}}\right)^{t-1} \\ &< M, \end{aligned}$$

where

$$M = 1 + \frac{1}{2^{k-1}} + \left(\frac{1}{2^{k-1}}\right)^2 + \left(\frac{1}{2^{k-1}}\right)^3 + \ldots = \sum_{n=1}^{\infty} \left(\frac{1}{2^{k-1}}\right)^{n-1}$$

is the sum of a convergent geometric series. ○

We now turn to some very simple properties of series. The proofs of the following three results are left as exercises.

THEOREM 3D. *The convergence or divergence of a series is unaffected if a finite number of terms are inserted, deleted or altered.*

THEOREM 3E. *Suppose that*

$$\sum_{n=1}^{\infty} z_n = s \qquad \text{and} \qquad \sum_{n=1}^{\infty} w_n = t.$$

Then for every real numbers $a, b \in \mathbb{R}$, we have

$$\sum_{n=1}^{\infty} (az_n + bw_n) = as + bt.$$

THEOREM 3F. *Suppose that the series (1) is convergent. Then $z_n \to 0$ as $n \to \infty$.*

REMARK. The converse of Theorem 3F is not true. For example, let $z_n = 1/n$. Clearly $z_n \to 0$ as $n \to \infty$. Note that the series (1) is not convergent in this case, in view of Theorem 3C.

3.2. Real Series

We first summarize the main idea in the proof of Theorem 3C.

THEOREM 3G. *Suppose that $x_n \geq 0$ for every $n \in \mathbb{N}$. Then the series*

$$\sum_{n=1}^{\infty} x_n$$

either converges to the supremum of the partial sums, or diverges to ∞.

PROOF. The partial sums form an increasing sequence. The result follows from Theorem 2E. ○

Very often, we can study the convergence or divergence of a series by comparing it with another series. We shall first of all study this phenomenon in the special case of series with non-negative terms.

THEOREM 3H. (COMPARISON TEST FOR SERIES WITH NON-NEGATIVE TERMS) *Let C be a positive constant independent of $n \in \mathbb{N}$. Suppose that for all sufficiently large natural numbers $n \in \mathbb{N}$, the inequalities $u_n \geq 0$, $v_n \geq 0$ and $u_n \leq Cv_n$ hold.*

(a) If $\displaystyle\sum_{n=1}^{\infty} v_n$ is convergent, then $\displaystyle\sum_{n=1}^{\infty} u_n$ is convergent.

(b) If $\displaystyle\sum_{n=1}^{\infty} u_n$ is divergent, then $\displaystyle\sum_{n=1}^{\infty} v_n$ is divergent.

PROOF. Note that (a) and (b) are equivalent, so we shall only prove (a). We shall use the General principle of convergence for series. Since the series

$$\sum_{n=1}^{\infty} v_n$$

is convergent, it follows that, given any $\epsilon > 0$, there exists N_0 such that for every natural number $n > N_0$, the three given inequalities hold, and

$$\sum_{n=N+1}^{M} v_n < \frac{\epsilon}{C} \qquad \text{whenever } M > N \geq N_0,$$

so that

$$\sum_{n=N+1}^{M} u_n < \epsilon \qquad \text{whenever } M > N \geq N_0.$$

The convergence of the series

$$\sum_{n=1}^{\infty} u_n$$

now follows from the General principle of convergence for series. ○

EXAMPLE 3.2.1. Suppose that $p \in \mathbb{Q}$ and $0 < a < 1$. We shall prove that the series

$$\sum_{n=1}^{\infty} n^p a^n \tag{3}$$

is convergent. Using the Ratio test for sequences, we can show that the sequence $n^{p+2}a^n \to 0$ as $n \to \infty$. It follows that for all sufficiently large natural numbers $n \in \mathbb{N}$, we have $n^{p+2}a^n < 1$, so that $n^p a^n < n^{-2}$. This last inequality allows us to compare the series (3) with the convergent harmonic series

$$\sum_{n=1}^{\infty} n^{-2}.$$

We now investigate series where the terms can be negative as well as non-negative real numbers. There is then the possibility of cancellation among terms. We first study a simple example.

EXAMPLE 3.2.2. Recall that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

is divergent. Let us now consider the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots. \tag{4}$$

Denote the partial sums by

$$s_N = \sum_{n=1}^{N} (-1)^{n-1} \frac{1}{n}.$$

Then it is not too difficult to see that for every $m \in \mathbb{N}$, we have

$$s_1 \geq s_3 \geq s_5 \geq \ldots \geq s_{2m-1} \geq s_{2m} \geq \ldots \geq s_6 \geq s_4 \geq s_2.$$

It follows that the sequence $s_1, s_3, s_5, \ldots$ is decreasing and bounded below by s_2, while the sequence $s_2, s_4, s_6, \ldots$ is increasing and bounded above by s_1. So both sequences converge. Note also that

$$s_{2m-1} - s_{2m} = \frac{1}{2m} \to 0$$

as $m \to \infty$, so that the two sequences converge to the same limit. This means that the sequence s_N converges as $N \to \infty$, so that the series (4) is convergent.

We now state and establish the result in general.

THEOREM 3J. (ALTERNATING SERIES TEST) *Suppose that*
(a) $a_n > 0$ for every $n \in \mathbb{N}$;
(b) a_n is a decreasing sequence; and
(c) $a_n \to 0$ as $n \to \infty$.
Then the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} a_n$$

is convergent.

PROOF. Consider the sequence of partial sums

$$s_N = \sum_{n=1}^{N} (-1)^{n-1} a_n.$$

In view of conditions (a) and (b), it is not too difficult to see that for every $m \in \mathbb{N}$, we have

$$s_1 \geq s_3 \geq s_5 \geq \ldots \geq s_{2m-1} \geq s_{2m} \geq \ldots \geq s_6 \geq s_4 \geq s_2.$$

It follows that the sequence $s_1, s_3, s_5, \ldots$ is decreasing and bounded below by s_2, while the sequence $s_2, s_4, s_6, \ldots$ is increasing and bounded above by s_1. So both sequences converge. Note also that in view of condition (c), we have

$$s_{2m-1} - s_{2m} = a_{2m} \to 0$$

as $m \to \infty$, so that the two sequences converge to the same limit. Hence the sequence s_N converges as $N \to \infty$. ○

EXAMPLE 3.2.3. The logarithmic series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$$

is convergent (with sum $\log 2$) if $x = 1$ and divergent if $x = -1$.

3.3. Complex Series

THEOREM 3K. *Suppose that $z_n \in \mathbb{C}$ for every $n \in \mathbb{N}$. If the series*

$$\sum_{n=1}^{\infty} |z_n| \tag{5}$$

is convergent, then the series

$$\sum_{n=1}^{\infty} z_n \tag{6}$$

is convergent. Furthermore, we have

$$\left| \sum_{n=1}^{\infty} z_n \right| \leq \sum_{n=1}^{\infty} |z_n|.$$

We shall give two proofs of this result. The first proof uses the General principle of convergence, while the second one relies on considering real and imaginary parts of the terms z_n and then studying the non-negative and negative parts of the real sequences that arise.

FIRST PROOF OF THEOREM 3K. Since the series (5) is convergent, it follows from the General principle of convergence for series that, given any $\epsilon > 0$, there exists a number N_0 such that

$$\sum_{n=N+1}^{M} |z_n| < \epsilon \qquad \text{whenever } M > N \geq N_0.$$

By the Triangle inequality, we have

$$\left|\sum_{n=N+1}^{M} z_n\right| \le \sum_{n=N+1}^{M} |z_n| < \epsilon \qquad \text{whenever } M > N \ge N_0.$$

It follows from the General principle of convergence for series that the series (6) is convergent. Note next that the sequence

$$T_N = \sum_{n=1}^{N} |z_n| - \left|\sum_{n=1}^{N} z_n\right|$$

is a non-negative convergent sequence as $N \to \infty$, in view of the Triangle inequality. It follows that

$$\lim_{N\to\infty} T_N = \sum_{n=1}^{\infty} |z_n| - \left|\sum_{n=1}^{\infty} z_n\right| \qquad \text{and} \qquad \lim_{N\to\infty} T_N \ge 0.$$

This completes the proof. ◯

Second Proof of Theorem 3K. Assume first of all that the first part of Theorem 3K holds for the special case when the sequence z_n is replaced by a real sequence u_n. Then for $z_n \in \mathbb{C}$, we write $z_n = x_n + \mathrm{i}y_n$, where $x_n, y_n \in \mathbb{R}$. Since the series (5) is convergent, the inequalities $|x_n| \le |z_n|$ and $|y_n| \le |z_n|$ enable us to use the Comparison test to conclude that the two series

$$\sum_{n=1}^{\infty} |x_n| \qquad \text{and} \qquad \sum_{n=1}^{\infty} |y_n|$$

are convergent, and so it follows from the special case of the first part of Theorem 3K that the series

$$\sum_{n=1}^{\infty} x_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} y_n$$

are convergent. The convergence of the series (6) now follows from Theorem 3E.

To show that the first part of Theorem 3K holds for real sequences u_n, note that for every $n \in \mathbb{N}$, we clearly have $u_n = u_n^+ - u_n^-$, where

$$u_n^+ = \begin{cases} u_n & \text{if } u_n \ge 0, \\ 0 & \text{if } u_n < 0, \end{cases}$$

and

$$u_n^- = \begin{cases} 0 & \text{if } u_n \ge 0, \\ -u_n & \text{if } u_n < 0. \end{cases}$$

Furthermore, $0 \le u_n^+ \le |u_n|$ and $0 \le u_n^- \le |u_n|$ for every $n \in \mathbb{N}$. If the series

$$\sum_{n=1}^{\infty} |u_n|$$

is convergent, then it follows from the Comparison test that the series

$$\sum_{n=1}^{\infty} u_n^+ \qquad \text{and} \qquad \sum_{n=1}^{\infty} u_n^-$$

are both convergent. The convergence of the series

$$\sum_{n=1}^{\infty} u_n = \sum_{n=1}^{\infty} (u_n^+ - u_n^-)$$

now follows from Theorem 3E.

The second part of Theorem 3K is proved in the same way as before. ◯

Definition. A series $\sum_{n=1}^{\infty} z_n$ is said to be absolutely convergent if the series $\sum_{n=1}^{\infty} |z_n|$ is convergent.

Remark. Theorem 3K states that every absolutely convergent series is convergent.

The Comparison test can now be stated in a much stronger form.

THEOREM 3L. (COMPARISON TEST) *Let C be a positive constant independent of $n \in \mathbb{N}$. Suppose that for all sufficiently large natural numbers $n \in \mathbb{N}$, the inequality $|z_n| \le Cv_n$ holds. Suppose further that the real series*

$$\sum_{n=1}^{\infty} v_n$$

is convergent. Then the series

$$\sum_{n=1}^{\infty} z_n$$

is absolutely convergent.

Much of the study of convergence of series is underpinned by our ability to compare a given series with an artificially constructed series. Two examples of this technique are given by the two tests below.

THEOREM 3M. (RATIO TEST) *Suppose that the sequence z_n satisfies*

$$\left| \frac{z_{n+1}}{z_n} \right| \to \ell \qquad \text{as } n \to \infty. \tag{7}$$

Then the series

$$\sum_{n=1}^{\infty} z_n \tag{8}$$

is absolutely convergent if $\ell < 1$ and divergent if $\ell > 1$.

Proof. Suppose first of all that $\ell < 1$. Let $L = \frac{1}{2}(1+\ell)$. Clearly $\ell < L < 1$. Since (7) holds, there exists an integer N such that

$$\left| \frac{z_{n+1}}{z_n} \right| < L \qquad \text{whenever } n \ge N.$$

It follows that

$$|z_n| < \frac{|z_N|}{L^N} L^n \qquad \text{whenever } n > N.$$

On the other hand, the geometric series

$$\sum_{n=1}^{\infty} L^n$$

is convergent. It follows from Comparison test that the series (8) is absolutely convergent. Suppose next that $\ell > 1$. Then clearly $|z_n| \not\to 0$ as $n \to \infty$. The result follows from Theorem 3F. ○

THEOREM 3N. (ROOT TEST) *Suppose that the sequence z_n satisfies*

$$|z_n|^{1/n} \to \ell \qquad \text{as } n \to \infty. \tag{9}$$

Then the series

$$\sum_{n=1}^{\infty} z_n \tag{10}$$

is absolutely convergent if $\ell < 1$ and divergent if $\ell > 1$.

PROOF. Suppose first of all that $\ell < 1$. Let $L = \frac{1}{2}(1 + \ell)$. Clearly $\ell < L < 1$. Since (9) holds, there exists an integer N such that

$$|z_n|^{1/n} < L \qquad \text{whenever } n > N.$$

It follows that

$$|z_n| < L^n \qquad \text{whenever } n > N.$$

On the other hand, the geometric series

$$\sum_{n=1}^{\infty} L^n$$

is convergent. It follows from Comparison test that the series (10) is absolutely convergent. Suppose next that $\ell > 1$. Then clearly $|z_n| \not\to 0$ as $n \to \infty$. The result follows from Theorem 3F. ○

REMARK. No firm conclusion can be drawn in the two settings above if $\ell = 1$. In the case of the Ratio test, consider the two series

$$\sum_{n=1}^{\infty} \frac{1}{n} \qquad \text{and} \qquad \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

It is easy to show that $\ell = 1$ in both cases. Note from Theorem 3C that the first series is divergent while the second series is convergent.

We conclude this section by considering rearrangements of a given series. The following example is famous.

EXAMPLE 3.3.1. Recall that the series

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \ldots$$

is divergent. On the other hand, the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{1}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots$$

is convergent, in view of the Alternating series test. Let s be its sum, so that

$$s = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots.$$

We next rearrange the terms and consider the series

$$\begin{aligned}1 - \frac{1}{2} - \frac{1}{4} + \frac{1}{3} - \frac{1}{6} - \frac{1}{8} + \frac{1}{5} - \frac{1}{10} - \frac{1}{12} + \ldots \\ = \left(1 - \frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{3} - \frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{5} - \frac{1}{10} - \frac{1}{12}\right) + \ldots \\ = \left(\frac{1}{2} - \frac{1}{4}\right) + \left(\frac{1}{6} - \frac{1}{8}\right) + \left(\frac{1}{10} - \frac{1}{12}\right) + \ldots \\ = \frac{1}{2}\left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6}\right) = \frac{s}{2}.\end{aligned}$$

Note that no term has been omitted or inserted in the rearrangement. Note also that $s \neq 0$. But yet we end up with a different sum. The only possible explanation is that the convergence of the original and the rearranged series depend on cancellation between positive and negative terms. The difference therefore has to arise from the nature of such cancellation.

Suppose now that the convergence of a series does not depend on the cancellation between positive and negative terms. Then it is reasonable to ask whether any rearrangement of the terms may still alter the sum of the series.

THEOREM 3P. *Any rearrangement of an absolutely convergent series*

$$\sum_{n=1}^{\infty} z_n \tag{11}$$

does not alter its sum.

Proof. Assume first of all that Theorem 3P holds for the special case when the sequence z_n is replaced by a real sequence u_n. Then for $z_n \in \mathbb{C}$, we write $z_n = x_n + \mathrm{i}y_n$, where $x_n, y_n \in \mathbb{R}$. Since the series (11) is absolutely convergent, the inequalities $|x_n| \le |z_n|$ and $|y_n| \le |z_n|$ enable us to use the Comparison test to conclude that the two series

$$\sum_{n=1}^{\infty} x_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} y_n$$

are absolutely convergent, and so it follows from the special case of Theorem 3P that rearrangement does not alter their sums. It now follows from Theorem 3E that rearrangement does not alter the sum of the series (11).

To establish the special case of Theorem 3P, suppose that the real series

$$\sum_{n=1}^{\infty} u_n$$

is absolutely convergent, and that the sequence v_n is a rearrangement of the sequence u_n. We now define $u_n^+, u_n^-, v_n^+, v_n^-$ in the same way as in the second proof of Theorem 3K. Then v_n^+ is a rearrangement of u_n^+ and v_n^- is a rearrangement of u_n^-. Clearly the series

$$\sum_{n=1}^{\infty} u_n^+$$

is convergent. Also, the sequence

$$\sum_{n=1}^{N} v_n^+$$

is increasing and bounded above by

$$\sum_{n=1}^{\infty} u_n^+,$$

so that

$$\sum_{n=1}^{\infty} v_n^+ \le \sum_{n=1}^{\infty} u_n^+.$$

Arguing in the opposite way, we must have

$$\sum_{n=1}^{\infty} u_n^+ \le \sum_{n=1}^{\infty} v_n^+.$$

Hence

$$\sum_{n=1}^{\infty} v_n^+ = \sum_{n=1}^{\infty} u_n^+.$$

Similarly,

$$\sum_{n=1}^{\infty} v_n^- = \sum_{n=1}^{\infty} u_n^-.$$

It now follows that

$$\sum_{n=1}^{\infty} v_n = \sum_{n=1}^{\infty} v_n^+ - \sum_{n=1}^{\infty} v_n^- = \sum_{n=1}^{\infty} u_n^+ - \sum_{n=1}^{\infty} u_n^- = \sum_{n=1}^{\infty} u_n,$$

and the proof is complete. ○

3.4. Power Series

Suppose that $z \in \mathbb{C}$. A series of the form

$$\sum_{n=0}^{\infty} a_n z^n, \tag{12}$$

where the coefficients $a_n \in \mathbb{C}$ for every $n \in \mathbb{N} \cup \{0\}$, is called a power series in the variable z. Note that it is convenient here to start the series with $n = 0$.

In the first two examples below, the case $z = 0$ is obvious, while the Ratio test can be applied to study the case $z \neq 0$.

EXAMPLE 3.4.1. The exponential series

$$\sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is absolutely convergent for every $z \in \mathbb{C}$.

EXAMPLE 3.4.2. The logarithmic series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\frac{z^n}{n}$$

is absolutely convergent for every $z \in \mathbb{C}$ satisfying $|z| < 1$ and is divergent for every $z \in \mathbb{C}$ satisfying $|z| > 1$.

EXAMPLE 3.4.3. The series

$$\sum_{n=1}^{\infty} n!z^n$$

is divergent for every non-zero $z \in \mathbb{C}$. To see this, we use Theorem 3F, and note that for any fixed $z \neq 0$, the sequence $n!z^n$ does not converge to 0 as $n \to \infty$.

The purpose of this section is to establish the following important result.

THEOREM 3Q. (CONVERGENCE THEOREM FOR POWER SERIES) *For the power series* (12), *exactly one of the following holds:*

(a) The series is absolutely convergent for every $z \in \mathbb{C}$.
(b) There exists a positive real number R such that the series is absolutely convergent for every $z \in \mathbb{C}$ satisfying $|z| < R$ and is divergent for every $z \in \mathbb{C}$ satisfying $|z| > R$.
(c) The series is divergent for every non-zero $z \in \mathbb{C}$.

DEFINITION. The number R in Theorem 3Q is called the radius of convergence of the power series (12). We also say that the radius of convergence is 0 if case (c) occurs, and that the power series (12) has infinite radius of convergence if case (a) occurs.

REMARK. Note that Theorem 3Q does not indicate whether the power series is convergent if $|z| = R$.

A crucial step in the proof of Theorem 3Q is summarized by the result below.

THEOREM 3R. *Suppose that the series* (12) *is convergent for a particular value $z = z_0$. Then the series is absolutely convergent for every $z \in \mathbb{C}$ satisfying $|z| < |z_0|$.*

PROOF. Suppose that the series

$$\sum_{n=0}^{\infty} a_n z_0^n$$

is convergent. Then it follows from Theorem 3F that $a_n z_0^n \to 0$ as $n \to \infty$. Recall that any convergent sequence is bounded, so that there exists $M \in \mathbb{R}$ such that $|a_n z_0^n| \leq M$ for every $n \in \mathbb{N} \cup \{0\}$. For every $z \in \mathbb{C}$ satisfying $|z| < |z_0|$, we have

$$|a_n z^n| \leq M \left|\frac{z}{z_0}\right|^n$$

for every $n \in \mathbb{N} \cup \{0\}$. Note that $|z/z_0| < 1$. Hence the series (12) is absolutely convergent by comparison with the convergent geometric series

$$\sum_{n=0}^{\infty} \left|\frac{z}{z_0}\right|^n.$$

This completes the proof. ◯

Proof of Theorem 3Q. Consider the set

$$S = \{x \geq 0 : \text{the series (12) converges}\}.$$

Clearly S contains the number 0, and is therefore non-empty. Exactly one of the following three cases applies:

(i) If S is not bounded above, then for every $z \in \mathbb{C}$, we can choose $x_0 \in S$ such that $|z| < x_0$. Since the series (12) is convergent at x_0, it follows from Theorem 3R that the series (12) is absolutely convergent at z.

(ii) Suppose that S is bounded above with supremum $R > 0$. For every $z \in \mathbb{C}$ satisfying $|z| < R$, we can choose $x_0 \in S$ such that $|z| < x_0$. Since the series (12) is convergent at x_0, it follows from Theorem 3R that the series (12) is absolutely convergent at z. On the other hand, for every $z \in \mathbb{C}$ satisfying $|z| > R$, we can choose $x_0 > R$ such that $|z| > x_0$. If the series (12) is convergent at z, then it follows from Theorem 3R that the series (12) is absolutely convergent at x_0, so that $x_0 \in S$, clearly a contradiction. Hence the series (12) must be divergent at z.

(iii) If $S = \{0\}$, then for every non-zero $z \in \mathbb{C}$, we can choose $x_0 > 0$ such that $|z| > x_0$. If the series (12) is convergent at z, then it follows from Theorem 3R that the series (12) is absolutely convergent at x_0, a contradiction. Hence the series (12) must be divergent at z. ○

3.5. Multiplication of Series

Multiplication of two series is not always a straightforward operation, in the sense that the product series may be affected by the order of the terms. The purpose of this section is to show that we need not worry if the series involved are absolutely convergent.

THEOREM 3S. *Suppose that the series*

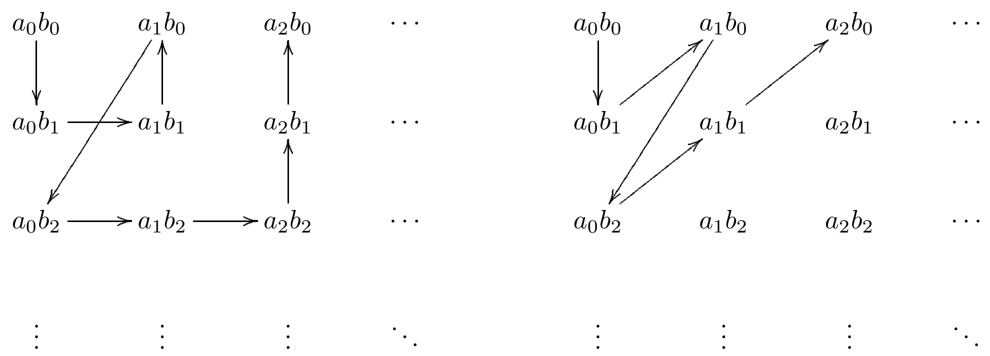
$$\sum_{n=0}^{\infty} a_n \qquad \textit{and} \qquad \sum_{n=0}^{\infty} b_n$$

are absolutely convergent, and converge to sums a and b respectively. Then the series

$$\sum a_i b_j, \tag{13}$$

consisting of the products, in any order, of every term of the first series by every term of the second series, is absolutely convergent, and converges to the sum ab.

Proof. The products of pairs of terms can be arranged in a doubly infinite array.



The sum of all these terms can be arranged as a single series. Two such ways are indicated above. We have summation by squares on the left, and diagonal summation on the right. No matter in what order the terms are arranged, the series

$$\sum |a_i b_j|$$

is a series of non-negative terms and clearly does not exceed

$$\left(\sum_{n=0}^{\infty} |a_n|\right)\left(\sum_{n=0}^{\infty} |b_n|\right).$$

It follows that the series (13) is absolutely convergent. In view of Theorem 3P, the sum is independent of the order of the arrangement of the terms. Since

$$\left(\sum_{n=0}^{N} a_n\right)\left(\sum_{n=0}^{N} b_n\right) \to ab \qquad \text{as } N \to \infty,$$

the sum must be ab. ○

THEOREM 3T. (CAUCHY PRODUCT) *Suppose that the series*

$$\sum_{n=0}^{\infty} a_n \qquad \textit{and} \qquad \sum_{n=0}^{\infty} b_n$$

are absolutely convergent, and converge to sums a and b respectively. Then the series

$$\sum_{n=0}^{\infty} c_n,$$

where

$$c_n = \sum_{r=0}^{n} a_r b_{n-r} \qquad \textit{for every } n \in \mathbb{N} \cup \{0\},$$

is absolutely convergent, and converges to the sum ab.

PROOF. This is simply using diagonal summation in Theorem 3S. ○

The Cauchy product is useful in establishing the following result on the exponential series.

THEOREM 3U. *The series*

$$E(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

is absolutely convergent for every $z \in \mathbb{C}$. Furthermore, for every $z_1, z_2 \in \mathbb{C}$, we have

$$E(z_1)E(z_2) = E(z_1 + z_2).$$

PROOF. The first part of the theorem is trivial for $z = 0$, and can be proved by using the Ratio test for $z \neq 0$. To prove the second part, note that

$$\frac{(z_1+z_2)^n}{n!} = \frac{1}{n!}\sum_{r=0}^{n} \frac{n!}{r!(n-r)!} z_1^r z_2^{n-r} = \sum_{r=0}^{n} \left(\frac{z_1^r}{r!}\right)\left(\frac{z_2^{n-r}}{(n-r)!}\right).$$

The result now follows from Theorem 3T. ○

Problems for Chapter 3

1. Let $a_n = -\dfrac{1}{n}$ if 3 divides n, and $a_n = \dfrac{1}{n}$ otherwise. By considering the sequence of partial sums s_{3N}, show that the series $\displaystyle\sum_{n=1}^{\infty} a_n$ is divergent.

2. For each of the following series, discuss whether the series is convergent or divergent, and justify your assertion:

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{n}{n^2+5n-3}$ b) $\displaystyle\sum_{n=1}^{\infty} (-1)^n(\sqrt{n+1}-\sqrt{n})$ c) $\displaystyle\sum_{n=1}^{\infty} \frac{(n!)^2}{(2n)!}$

   d) $\displaystyle\sum_{n=1}^{\infty} (n!)^{1/n}$ e) $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}\sin\frac{n\pi}{2}$ f) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{n}{n+1}\right)^{n^2}$

   g) $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left(1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}\right)$

3. For each of the following series, determine the values of $x \in \mathbb{R}$ for which the series is convergent, and justify your assertion:

   a) $\displaystyle\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}$ b) $\displaystyle\sum_{n=1}^{\infty} \sin nx$ c) $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1}\frac{x^n}{n}$ d) $\displaystyle\sum_{n=1}^{\infty} \frac{n^x}{n^2-2}$

4. Suppose that $u_n \geq 0$ and $v_n \geq 0$ for every $n \in \mathbb{N}$. Suppose further that $u_n/v_n \to 2$ as $n \to \infty$. Show that the series
$$\sum_{n=1}^{\infty} u_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} v_n$$
are either both convergent or both divergent.

5. a) Suppose that the real series $\displaystyle\sum_{n=1}^{\infty} a_n$ and $\displaystyle\sum_{n=1}^{\infty} b_n$ are both convergent. Suppose further that $a_n \geq 0$ and $b_n \geq 0$ for every $n \in \mathbb{N}$. Prove that the series $\displaystyle\sum_{n=1}^{\infty} a_n b_n$ is convergent.

   b) Discuss also the case when the terms a_n and b_n can be negative.

6. For every $n \in \mathbb{N}$, let $a_n = \dfrac{1}{\sqrt{n}} + \dfrac{(-1)^{n+1}}{n}$.

   a) Show that $a_n > 0$ for every $n \in \mathbb{N}$, and that $a_n \to 0$ as $n \to \infty$.

   b) Show that the series $\displaystyle\sum_{n=1}^{\infty} (-1)^n a_n$ is divergent.

   c) Comment on the result.

7. For each of the following series, determine the values of $z \in \mathbb{C}$ for which the series is convergent, and justify your assertion:

   a) $\displaystyle\sum_{n=1}^{\infty} z^{n^2}$ b) $\displaystyle\sum_{n=1}^{\infty} n!z^n$ c) $\displaystyle\sum_{n=1}^{\infty} n!z^{n!}$

8. Suppose that $\dfrac{22}{7} \leq |a_n| \leq 100$ for every $n \in \mathbb{N}\cup\{0\}$. Discuss the radius of convergence of the power series $\displaystyle\sum_{n=0}^{\infty} a_n z^n$, and justify your assertion.