FUNDAMENTALS OF ANALYSIS

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Chapter 4

FUNCTIONS AND CONTINUITY

4.1. Limits of Functions

We begin by studying the behaviour of a function $f(x)$ as $x \to +\infty$. Corresponding to the definition of the limit of a real sequence, we have the following direct analogue for real valued functions of a real variable. In this chapter, all functions $f(x)$ are assumed to be real valued and are defined on $\mathbb{R}$ or suitable subsets of $\mathbb{R}$.

Definition. We say that $f(x) \to L$ as $x \to +\infty$, or

$$\lim_{x\to+\infty} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $D > 0$ such that $|f(x) - L| < \epsilon$ whenever $x > D$.

We can also study the behaviour of a function $f(x)$ as $x \to -\infty$. Corresponding to the above, we have the following obvious analogue.

Definition. We say that $f(x) \to L$ as $x \to -\infty$, or

$$\lim_{x\to-\infty} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $D > 0$ such that $|f(x) - L| < \epsilon$ whenever $x < -D$.

It is not difficult to see that we can establish suitable analogues of Theorems 2A, 2C and 2D concerning the uniqueness of limits, the arithmetic of limits and the Squeezing principle respectively.

While the natural numbers are discrete, the real number line is a continuous object. We can therefore also study the behaviour of a function $f(x)$ as x gets close to a given real number a.

DEFINITION. We say that $f(x) \to L$ as $x \to a$, or

$$\lim_{x\to a} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < |x - a| < \delta$.

REMARK. The restriction $|x - a| > 0$ is to omit discussion of the situation when $x = a$. After all, we are only interested in those x which are close to a but not equal to a.

Much of the theory of limits of sequences can be translated to this new setting of limits of functions as $x \to a$, courtesy of the result below.

THEOREM 4A. *We have $f(x) \to L$ as $x \to a$ if and only if $f(x_n) \to L$ as $n \to \infty$ for every sequence x_n of real numbers such that $x_n \neq a$ for any $n \in \mathbb{N}$ and $x_n \to a$ as $n \to \infty$.*

PROOF. Suppose first of all that $f(x) \to L$ as $x \to a$. Then given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - L| < \epsilon \qquad \text{whenever } 0 < |x - a| < \delta.$$

Let x_n be any sequence of real numbers such that $x_n \neq a$ for any $n \in \mathbb{N}$ and $x_n \to a$ as $n \to \infty$. Then there exists $N \in \mathbb{R}$ such that

$$0 \neq |x_n - a| < \delta \qquad \text{whenever } n > N.$$

Hence

$$|f(x_n) - L| < \epsilon \qquad \text{whenever } n > N.$$

This shows that $f(x_n) \to L$ as $n \to \infty$.

Suppose next that $f(x) \not\to L$ as $x \to a$. Then there exists $\epsilon > 0$ such that for every $n \in \mathbb{N}$, there exists x_n such that

$$0 < |x_n - a| < \frac{1}{n} \qquad \text{and} \qquad |f(x_n) - L| \geq \epsilon.$$

Clearly $x_n \neq a$ for any $n \in \mathbb{N}$ and $x_n \to a$ as $n \to \infty$. However, it is not difficult to see that $f(x_n) \not\to L$ as $n \to \infty$. ◯

Using Theorem 4A, we can immediately establish the following three results which are the analogues of Theorems 2A, 2C and 2D respectively.

THEOREM 4B. *The limit of a function as $x \to a$ is unique if it exists.*

THEOREM 4C. *Suppose that the functions $f(x) \to L$ and $g(x) \to M$ as $x \to a$. Then*
(a) $f(x) + g(x) \to L + M$ as $x \to a$;
(b) $f(x)g(x) \to LM$ as $x \to a$; and
(c) if $M \neq 0$, then $f(x)/g(x) \to L/M$ as $x \to a$.

THEOREM 4D. *Suppose that $g(x) \leq f(x) \leq h(x)$ for every $x \neq a$ in some open interval containing a. Suppose further that $g(x) \to L$ and $h(x) \to L$ as $x \to a$. Then $f(x) \to L$ as $x \to a$.*

A similar theory can be established on one-sided limits.

Definition. We say that $f(x) \to L$ as $x \to a+$, or

$$\lim_{x \to a+} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < x - a < \delta$. In this case, L is called the right-hand limit.

Definition. We say that $f(x) \to L$ as $x \to a-$, or

$$\lim_{x \to a-} f(x) = L,$$

if, for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $0 < a - x < \delta$. In this case, L is called the left-hand limit.

It is very easy to deduce the following result.

THEOREM 4E. *We have*

$$\lim_{x \to a} f(x) = L \qquad \textit{if and only if} \qquad \lim_{x \to a-} f(x) = \lim_{x \to a+} f(x) = L.$$

It is not difficult to formulate suitable analogues of the arithmetic of limits and the Squeezing principle. Their precise statements are left as exercises.

Definition. We say that a function $f(x)$ is continuous at $x = a$ if $f(x) \to f(a)$ as $x \to a$; in other words, if

$$\lim_{x \to a} f(x) = f(a).$$

Since continuity is defined in terms of limits, we immediately have the following consequences of Theorem 4C.

THEOREM 4F. *Suppose that the functions $f(x)$ and $g(x)$ are continuous at $x = a$. Then*
(a) $f(x) + g(x)$ is continuous at $x = a$;
(b) $f(x)g(x)$ is continuous at $x = a$; and
(c) if $g(a) \neq 0$, then $f(x)/g(x)$ is continuous at $x = a$.

4.2. Continuity in Intervals

Definition. Suppose that $A, B \in \mathbb{R}$ with $A < B$. We say that a function $f(x)$ is continuous in the open interval (A, B) if $f(x)$ is continuous at $x = a$ for every $a \in (A, B)$.

To formulate a suitable definition for continuity in a closed interval, we consider first an example.

Example 4.2.1. Consider the function

$$f(x) = \begin{cases} 1 & \text{if } x \geq 0, \\ 0 & \text{if } x < 0. \end{cases}$$

It is clear that this function is not continuous at $x = 0$, since

$$\lim_{x \to 0-} f(x) = 0 \qquad \text{and} \qquad \lim_{x \to 0+} f(x) = 1.$$

However, let us investigate the behaviour of the function in the closed interval $[0,1]$. It is clear that $f(x)$ is continuous at $x=a$ for every $a\in(0,1)$. Furthermore, we have

$$\lim_{x\to 0+} f(x)=f(0) \qquad \text{and} \qquad \lim_{x\to 1-} f(x)=f(1).$$

This example leads us to conclude that it is not appropriate to insist on continuity of the function at the end-points of the closed interval, and that a more suitable requirement is one-sided continuity instead.

DEFINITION. Suppose that $A,B\in\mathbb{R}$ with $A<B$. We say that a function $f(x)$ is continuous in the closed interval $[A,B]$ if $f(x)$ is continuous in the open interval (A,B) and if

$$\lim_{x\to A+} f(x)=f(A) \qquad \text{and} \qquad \lim_{x\to B-} f(x)=f(B).$$

REMARK. It follows that for continuity of a function in a closed interval, we need right-hand continuity of the function at the left-hand end-point of the interval, left-hand continuity of the function at the right-hand end-point of the interval, and continuity at every point in between.

Observe that so far in our discussion in this chapter, there has been no analogue of Theorem 2B concerning boundedness.

4.3. Continuity in Closed Intervals

DEFINITION. Suppose that a function $f(x)$ is defined on an interval $I\subseteq\mathbb{R}$. We say that $f(x)$ is bounded above on I if there exists a real number $K\in\mathbb{R}$ such that $f(x)\le K$ for every $x\in I$, and that $f(x)$ is bounded below on I if there exists a real number $k\in\mathbb{R}$ such that $f(x)\ge k$ for every $x\in I$. Furthermore, we say that $f(x)$ is bounded on I if it is bounded above and bounded below on I.

The following can be considered an analogue of Theorem 2B.

THEOREM 4G. *Suppose that a function $f(x)$ is continuous in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ with $A<B$. Then $f(x)$ is bounded on $[A,B]$.*

PROOF. Suppose on the contrary that $f(x)$ is not bounded on $[A,B]$. Then it is either not bounded above on $[A,B]$ or not bounded below on $[A,B]$, or both. By considering the function $-f(x)$ if necessary, we may assume, without loss of generality, that $f(x)$ is not bounded above on $[A,B]$. Then for every $n\in\mathbb{N}$, there exists $x_n\in[A,B]$ such that $f(x_n)>n$. The real sequence x_n is clearly bounded. It follows from Theorem 2K that x_n has a convergent subsequence x_{n_p}, say. Suppose that $x_{n_p}\to c$ as $p\to\infty$. Clearly $c\in[A,B]$. Suppose first of all that $c\in(A,B)$. Since $f(x)$ is continuous at $x=c$, it follows from Theorem 4A that $f(x_{n_p})\to f(c)$ as $p\to\infty$. But this is a contradiction, since the sequence $f(x_{n_p})$ satisfies $f(x_{n_p})>n_p\ge p$ for every $p\in\mathbb{N}$, and so is not bounded, and hence not convergent in view of Theorem 2B. If $c=A$ or $c=B$, then there is only one-sided continuity at $x=c$, and the proof has to be slightly modified. ◯

In fact, we can establish more.

THEOREM 4H. (MAX-MIN THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ with $A<B$. Then there exist real numbers $x_1,x_2\in[A,B]$ such that $f(x_1)\le f(x)\le f(x_2)$ for every $x\in[A,B]$. In other words, the function $f(x)$ attains a maximum value and a minimum value in the closed interval $[A,B]$.*

PROOF. We shall only establish the existence of the real number $x_2 \in [A, B]$, as the existence of the real number $x_1 \in [A, B]$ can be established by repeating the argument here on the function $-f(x)$. Note first of all that it follows from Theorem 4G that the set

$$S = \{f(x) : x \in [A, B]\}$$

is bounded above. Let $M = \sup S$. Then $f(x) \le M$ for every $x \in [A, B]$. Suppose on the contrary that there does not exist $x_2 \in [A, B]$ such that $f(x_2) = M$. Then $f(x) < M$ for every $x \in [A, B]$, and so it follows from Theorem 4F that the function

$$g(x) = \frac{1}{M - f(x)}$$

is continuous in the closed interval $[A, B]$, and is therefore bounded above on $[A, B]$ as a consequence of Theorem 4G. Suppose that $g(x) \le K$ for every $x \in [A, B]$. Since $g(x) > 0$ for every $x \in [A, B]$, we must have $K > 0$. But then the inequality $g(x) \le K$ gives the inequality

$$f(x) \le M - \frac{1}{K},$$

contradicting the assumption that $M = \sup S$. ○

THEOREM 4J. (INTERMEDIATE VALUE THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that the real numbers $x_1, x_2 \in [A, B]$ satisfy $f(x_1) \le f(x) \le f(x_2)$ for every $x \in [A, B]$. Then for every real number $y \in \mathbb{R}$ satisfying $f(x_1) \le y \le f(x_2)$, there exists a real number $x_0 \in [A, B]$ such that $f(x_0) = y$.*

PROOF. We may clearly suppose that $f(x_1) < y < f(x_2)$. By considering the function $-f(x)$ if necessary, we may further assume, without loss of generality, that $x_1 < x_2$. The idea of the proof is then to follow the graph of the function $f(x)$ from the point $(x_1, f(x_1))$ to the point $(x_2, f(x_2))$. This clearly touches the horizontal line at height y at least once; the reader is advised to draw a picture. Our technique is then to trap the last occasion when this happens. Accordingly, we consider the set

$$T = \{x \in [x_1, x_2] : f(x) \le y\}.$$

This set is clearly bounded above. Let $x_0 = \sup T$. We shall show that $f(x_0) = y$. Suppose on the contrary that $f(x_0) \ne y$. Then exactly one of the following two cases applies:

(i) We have $f(x_0) > y$. In this case, let $\epsilon = f(x_0) - y > 0$. Since $f(x)$ is continuous at $x = x_0$, it follows that there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. This implies that $f(x) > y$ for every real number $x \in (x_0 - \delta, x_0 + \delta)$, so that $x_0 - \delta$ is an upper bound of T, contradicting the assumption that $x_0 = \sup T$.

(ii) We have $f(x_0) < y$. In this case, let $\epsilon = y - f(x_0) > 0$. Since $f(x)$ is continuous at $x = x_0$, it follows that there exists $\delta > 0$ such that $|f(x) - f(x_0)| < \epsilon$ whenever $|x - x_0| < \delta$. This implies that $f(x) < y$ for every real number $x \in (x_0 - \delta, x_0 + \delta)$, so that x_0 cannot be an upper bound of T, again contradicting the assumption that $x_0 = \sup T$. ○

REMARK. Suppose that the function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Then Theorems 4G, 4H and 4J together imply that the range

$$f([A, B]) = \{f(x) : x \in [A, B]\}$$

is a closed interval. In other words, a continuous real valued function of a real variable maps a closed interval to another closed interval.

Problems for Chapter 4

1. Consider the function
$$f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Prove that $f(x)$ is continuous at 0.

2. Consider the function
$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 1 - x & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$
   a) Prove that $f(x)$ is discontinuous everywhere except at $\frac{1}{2}$.
   b) Hence, or otherwise, find a bijection $g : [0, 1] \to [0, 1]$ which is discontinuous everywhere in $(0, 1)$.

3. Consider the function
$$f(x) = \begin{cases} \mathrm{e}^{-1/|x|} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$
Prove that $f(x)$ is continuous in $\mathbb{R}$.

4. A function $f : \mathbb{R} \to \mathbb{R}$ is continuous at every $x \in \mathbb{R}$, and satisfies $f(x) \to 0$ as $x \to +\infty$ as well as $f(x) \to 3$ as $x \to -\infty$. Prove that the range $f(\mathbb{R})$ is bounded.

5. Suppose that a function $f : [A, B] \to \mathbb{R}$ is continuous and strictly increasing in the closed interval $[A, B]$, so that $f(x_1) < f(x_2)$ whenever $A \leq x_1 < x_2 \leq B$. Suppose further that $f(A) = \alpha$ and $f(B) = \beta$.
   a) Explain why $\{f(x) : x \in [A, B]\} = [\alpha, \beta]$.
   b) Show that for every $y \in [\alpha, \beta]$, there exists a unique $x \in [A, B]$ such that $f(x) = y$.
   c) Show that the function $g : [\alpha, \beta] \to [A, B]$, defined for every $y \in [\alpha, \beta]$ by $g(y) = x$, where $x \in [A, B]$ is uniquely determined in part (b) by $f(x) = y$, is strictly increasing and continuous in the closed interval $[\alpha, \beta]$.