FUNDAMENTALS OF ANALYSIS

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Chapter 5

DIFFERENTIATION

5.1. Introduction

We begin by recalling the familiar definition of differentiability.

Definition. We say that a function $f(x)$ is differentiable at $x = a$ if the limit

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

exists. In this case, the limit is denoted by $f'(a)$ and called the derivative of $f(x)$ at $x = a$.

Example 5.1.1. Consider the function $f(x) = c$, where $c \in \mathbb{R}$ is a constant. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = 0 \to 0$$

as $x \to a$. It follows that $f'(a) = 0$ for every $a \in \mathbb{R}$.

Example 5.1.2. Consider the function $f(x) = x$. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = 1 \to 1$$

as $x \to a$. It follows that $f'(a) = 1$ for every $a \in \mathbb{R}$.

Example 5.1.3. Consider the function $f(x) = x^n$, where $n \geq 2$ is an integer. For every $a \in \mathbb{R}$, we have

$$\frac{f(x) - f(a)}{x - a} = \frac{x^n - a^n}{x - a} = x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \ldots + x^2a^{n-3} + xa^{n-2} + a^{n-1} \to na^{n-1}$$

as $x \to a$. It follows that $f'(a) = na^{n-1}$ for every $a \in \mathbb{R}$.

EXAMPLE 5.1.4. Consider the function $f(x) = \sqrt{x}$. For every positive $a \in \mathbb{R}$, we have

$$\frac{f(x)-f(a)}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{x-a} = \frac{\sqrt{x}-\sqrt{a}}{(\sqrt{x}-\sqrt{a})(\sqrt{x}+\sqrt{a})} = \frac{1}{\sqrt{x}+\sqrt{a}} \to \frac{1}{2\sqrt{a}}$$

as $x \to a$. It follows that $f'(a) = 1/2\sqrt{a}$ for every positive $a \in \mathbb{R}$.

EXAMPLE 5.1.5. Consider the function $f(x) = \sin x$. For every $a \in \mathbb{R}$, we have

$$\frac{f(x)-f(a)}{x-a} = \frac{\sin x - \sin a}{x-a} = \frac{2\cos\frac{1}{2}(x+a)\sin\frac{1}{2}(x-a)}{x-a} = \frac{\sin\frac{1}{2}(x-a)}{\frac{1}{2}(x-a)}\cos\frac{1}{2}(x+a) \to \cos a$$

as $x \to a$. It follows that $f'(a) = \cos a$ for every $a \in \mathbb{R}$.

EXAMPLE 5.1.6. Consider the function $f(x) = \cos x$. For every $a \in \mathbb{R}$, we have

$$\frac{f(x)-f(a)}{x-a} = \frac{\cos x - \cos a}{x-a} = -\frac{2\sin\frac{1}{2}(x+a)\sin\frac{1}{2}(x-a)}{x-a} = -\frac{\sin\frac{1}{2}(x-a)}{\frac{1}{2}(x-a)}\sin\frac{1}{2}(x+a) \to -\sin a$$

as $x \to a$. It follows that $f'(a) = -\sin a$ for every $a \in \mathbb{R}$.

EXAMPLE 5.1.7. Consider the function $f(x) = x^{1/3}$. For every non-zero $a \in \mathbb{R}$, we have

$$\frac{f(x)-f(a)}{x-a} = \frac{x^{1/3}-a^{1/3}}{x-a} = \frac{1}{x^{2/3}+x^{1/3}a^{1/3}+a^{2/3}} \to \frac{1}{3a^{2/3}}$$

as $x \to a$. It follows that $f'(a) = \frac{1}{3}a^{-2/3}$ for every non-zero $a \in \mathbb{R}$. On the other hand, we note that

$$\frac{f(x)-f(0)}{x-0} = \frac{x^{1/3}}{x} = \frac{1}{x^{2/3}}$$

does not tend to a limit as $x \to 0$, so that the function $f(x)$ is not differentiable at $x = 0$.

Examples 5.1.3 and 5.1.7 above raise the question of determining derivatives of functions of the type $f(x) = x^n$, where n is a real number, not necessarily a positive integer. We state the following important result.

THEOREM 5A. *Suppose that $n \in \mathbb{Q}$ is a fixed rational number. Then for the function $f(x) = x^n$, we have $f'(a) = na^{n-1}$ for every $a \in \mathbb{R}$, except for*
(a) $a = 0$ and $n < 1$; or
(b) $a \le 0$ when $n = p/q$ in lowest terms with $p \in \mathbb{Z}$ and even $q \in \mathbb{N}$.

We shall leave the proof of this result until later in this section.

EXAMPLE 5.1.8. Consider the function

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}, \\ 0 & \text{if } x \in \mathbb{R} \setminus \mathbb{Q}. \end{cases}$$

For every $a \in \mathbb{R}$, it is not difficult to check that

$$\frac{f(x)-f(a)}{x-a}$$

does not tend to a limit as $x \to a$, so that the function $f(x)$ is differentiable nowhere.

EXAMPLE 5.1.9. Consider the function $f(x) = |x|$, so that

$$f(x) = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

For every non-zero $a \in \mathbb{R}$, it is not difficult to check that

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \begin{cases} 1 & \text{if } a > 0, \\ -1 & \text{if } a < 0, \end{cases}$$

so that $f'(a) = 1$ for every positive $a \in \mathbb{R}$ and $f'(a) = -1$ for every negative $a \in \mathbb{R}$. On the other hand, we note that

$$\frac{f(x) - f(0)}{x - 0}$$

does not tend to a limit as $x \to 0$, so that the function $f(x)$ is not differentiable at $x = 0$.

Suppose that a function $f(x)$ is differentiable at $x = a$. Then

$$\frac{f(x) - f(a)}{x - a} \to f'(a)$$

as $x \to a$. On the other hand, clearly the function $x - a \to 0$ as $x \to a$. By the product rule of limits, we have

$$f(x) - f(a) = \left(\frac{f(x) - f(a)}{x - a} \right) (x - a) \to 0$$

as $x \to a$. It follows that $f(x) \to f(a)$ as $x \to a$. We have therefore established the following result.

THEOREM 5B. *Suppose that a function $f(x)$ is differentiable at $x = a$. Then $f(x)$ is continuous at $x = a$.*

As is in the case of limits and continuity, we have the sum, product and quotient rules for derivatives. We shall establish the following result.

THEOREM 5C. *Suppose that the functions $f(x)$ and $g(x)$ are differentiable at $x = a$. Then*
(a) $f(x) + g(x)$ is differentiable at $x = a$;
(b) $f(x)g(x)$ is differentiable at $x = a$; and
(c) if $g(a) \neq 0$, then $f(x)/g(x)$ is differentiable at $x = a$.
Furthermore, we have
(a) $(f + g)'(a) = f'(a) + g'(a)$;
(b) $(fg)'(a) = f(a)g'(a) + f'(a)g(a)$; and
(c) $\displaystyle \left(\frac{f}{g} \right)' (a) = \frac{g(a)f'(a) - f(a)g'(a)}{g^2(a)}$.

PROOF. (a) Note that

$$\frac{(f(x) + g(x)) - (f(a) + g(a))}{x - a} = \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a}.$$

It follows from Theorem 4C that

$$\lim_{x \to a} \frac{(f(x) + g(x)) - (f(a) + g(a))}{x - a} = f'(a) + g'(a).$$

(b) Note that

$$\begin{aligned}\frac{f(x)g(x)-f(a)g(a)}{x-a}&=\frac{f(x)g(x)-f(x)g(a)+f(x)g(a)-f(a)g(a)}{x-a}\\&=f(x)\frac{g(x)-g(a)}{x-a}+g(a)\frac{f(x)-f(a)}{x-a}.\end{aligned}$$

In view of Theorem 5B, we clearly have $f(x)\to f(a)$ as $x\to a$. It follows from Theorem 4C that

$$\lim_{x\to a}\frac{f(x)g(x)-f(a)g(a)}{x-a}=f(a)g'(a)+g(a)f'(a).$$

(c) We shall first show that $1/g(x)$ is differentiable at $x=a$. Note that

$$\frac{(1/g(x))-(1/g(a))}{x-a}=-\frac{g(x)-g(a)}{x-a}\frac{1}{g(x)}\frac{1}{g(a)}.$$

In view of Theorem 5B, we clearly have $g(x)\to g(a)$ as $x\to a$. It follows from Theorem 4C that

$$\lim_{x\to a}\frac{(1/g(x))-(1/g(a))}{x-a}=-\frac{g'(a)}{g^2(a)}.$$

We now apply part (b) to $f(x)$ and $1/g(x)$ to get the desired result. ◯

Example 5.1.10. Consider the function $f(x)=\tan x$. We know that

$$\tan x=\frac{\sin x}{\cos x}.$$

It follows that for every $a\in\mathbb{R}$ such that $\cos a\neq 0$, we have, by the quotient rule, that

$$f'(a)=\frac{\cos^2 a+\sin^2 a}{\cos^2 a}=\frac{1}{\cos^2 a}=\sec^2 a.$$

Example 5.1.11. Consider the function $f(x)=\csc x$. We know that

$$\csc x=\frac{1}{\sin x}.$$

It follows that for every $a\in\mathbb{R}$ such that $\sin a\neq 0$, we have, by the quotient rule, that

$$f'(a)=\frac{0-\cos a}{\sin^2 a}=-\cot a\csc a.$$

Example 5.1.12. Consider the function

$$f(x)=\frac{x^3\sin x}{x^2+3}.$$

We can write $f(x)=g(x)/h(x)$, where $g(x)=x^3\sin x$ and $h(x)=x^2+3$. For every $a\in\mathbb{R}$, we have $g'(a)=a^3\cos a+3a^2\sin a$ and $h'(a)=2a$. It follows that

$$f'(a)=\frac{h(a)g'(a)-g(a)h'(a)}{h^2(a)}=\frac{(a^2+3)(a^3\cos a+3a^2\sin a)-2a^4\sin a}{(a^2+3)^2}.$$

From now on, we shall slightly abuse our notation, and simply refer to $f'(x)$ as the derivative of the function $f(x)$. We shall further write

$$y = f(x) \qquad \text{and} \qquad \frac{\mathrm{d}y}{\mathrm{d}x} = f'(x).$$

It follows, for example, that if we write

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{x}{\sin x}\right) = \frac{\sin x - x\cos x}{\sin^2 x},$$

then we mean that we are considering the function $f(x) = x/\sin x$, and that for every $a \in \mathbb{R}$ for which $\sin a \neq 0$, we have $f'(a) = (\sin a - a\cos a)/\sin^2 a$.

An important technique in differentiation is through the use of composite functions.

EXAMPLE 5.1.13. Let $y = (x^3+1)^2$. To calculate the derivative $\mathrm{d}y/\mathrm{d}x$, we can first of all write $y = x^6 + 2x^3 + 1$, and then differentiate to obtain

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 6x^5 + 6x^2 = 6x^2(x^3+1).$$

Let us look at this in a different way. We can write $y = u^2$, where $u = x^3 + 1$. Then

$$\frac{\mathrm{d}y}{\mathrm{d}u} = 2u \qquad \text{and} \qquad \frac{\mathrm{d}u}{\mathrm{d}x} = 3x^2.$$

Note that

$$\frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x} = 6ux^2 = 6x^2(x^3+1).$$

We therefore have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

THEOREM 5D. *Suppose that y is a differentiable function of u, and that u is a differentiable function of x. Then y is a differentiable function of x, and*

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u}\frac{\mathrm{d}u}{\mathrm{d}x}.$$

PROOF. Write $y = g(u)$, $u = f(x)$ and $b = f(a)$. Then $y = (g \circ f)(x)$. Note that

$$\frac{(g\circ f)(x) - (g\circ f)(a)}{x-a} = \frac{(g\circ f)(x) - (g\circ f)(a)}{f(x)-f(a)}\frac{f(x)-f(a)}{x-a} = \frac{g(u)-g(b)}{u-b}\frac{f(x)-f(a)}{x-a}.$$

Here it is tempting to deduce the conclusion immediately. However, it is possible that $u - b = 0$. To overcome this difficulty, let us introduce the function

$$G(u) = \begin{cases} \dfrac{g(u)-g(b)}{u-b} & \text{if } u \neq b, \\ g'(b) & \text{if } u = b. \end{cases}$$

Since $g(u)$ is differentiable at $u = b$, we have $G(u) \to g'(b)$ as $u \to b$. Furthermore, since $G(b) = g'(b)$, it follows that $G(u)$ is continuous at $u = b$. On the other hand, as $x \to a$, we have $u \to b$, so that $G(u) \to g'(b)$. Hence

$$G(u) \to g'(b) \qquad \text{as } x \to a.$$

Suppose now that $u \neq b$. Then we clearly have

$$\frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = G(u) \frac{f(x) - f(a)}{x - a}.$$

Note that this also holds when $u = b$, since both sides are equal to 0. It now follows that

$$\lim_{x \to a} \frac{(g \circ f)(x) - (g \circ f)(a)}{x - a} = g'(b) f'(a) = g'(f(a)) f'(a)$$

as required. ◯

DEFINITIONS.
(1) A function $f(x)$ is said to be strictly increasing in the closed interval $[A, B]$ if $f(x_1) < f(x_2)$ whenever $A \leq x_1 < x_2 \leq B$.
(2) A function $f(x)$ is said to be strictly decreasing in the closed interval $[A, B]$ if $f(x_1) > f(x_2)$ whenever $A \leq x_1 < x_2 \leq B$.

THEOREM 5E. *Suppose that a function $y = f(x)$ is continuous and strictly increasing in the closed interval $[A, B]$. Suppose further that $f(x)$ is differentiable at $x = a$ for some $a \in (A, B)$, with $f(a) = b$ and $f'(a) \neq 0$. Then the inverse function $x = g(y)$ is differentiable at $y = b$, with*

$$g'(b) = \frac{1}{f'(a)}.$$

PROOF. The existence of the continuous and strictly increasing inverse function is a consequence of Problem 5 for Chapter 4. Note next that

$$\frac{g(y) - g(b)}{y - b} = \frac{x - a}{f(x) - f(a)},$$

and that $x \to a$ as $y \to b$, a consequence of the continuity of the inverse function. ◯

PROOF OF THEOREM 5A. The case when n is a positive integer has been studied in Examples 5.1.2 and 5.1.3. The case when $n = 0$ and $a \neq 0$ has been studied in Example 5.1.1. Suppose next that n is a negative integer. Then $-n$ is a positive integer, and

$$\begin{aligned}
\frac{f(x) - f(a)}{x - a} &= \frac{1}{x - a} \left(\frac{1}{x^{-n}} - \frac{1}{a^{-n}} \right) = -\frac{x^{-n} - a^{-n}}{(x - a) x^{-n} a^{-n}} \\
&= -\frac{x^{-n-1} + x^{-n-2} a + x^{-n-3} a^2 + \ldots + x^2 a^{-n-3} + x a^{-n-2} + a^{-n-1}}{x^{-n} a^{-n}} \\
&\to \frac{n a^{-n-1}}{a^{-2n}} = n a^{n-1}
\end{aligned}$$

as $x \to a$, provided that $a \neq 0$. Suppose now that $n = p/q$ in lowest terms, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$, and where exceptions (a) and (b) do not hold. Then $y = x^n$ can be described by $y = u^p$ and $u = x^{1/q}$, so that $x = u^q$ in particular. By Theorems 5D and 5E, we have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \frac{\mathrm{d}u}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \Big/ \frac{\mathrm{d}x}{\mathrm{d}u} = \frac{p u^{p-1}}{q u^{q-1}} = \frac{p}{q} u^{p-q} = n x^{n-1}.$$

This completes the proof. ◯

EXAMPLE 5.1.14. Consider the function $f(x) = c^x$, where $c \in \mathbb{R}$ is a fixed positive real number. Then

$$\frac{f(x) - f(a)}{x - a} = \frac{c^x - c^a}{x - a} = c^a \frac{c^{x-a} - 1}{x - a} \to c^a \lim_{h \to 0} \frac{c^h - 1}{h}$$

as $x \to a$. In the special case when $c = \mathrm{e}$, we have

$$\lim_{h\to 0} \frac{c^h - 1}{h} = 1,$$

so that for the function $f(x) = \mathrm{e}^x$, we have

$$\frac{f(x) - f(a)}{x - a} \to \mathrm{e}^a$$

as $x \to a$. Hence $f'(a) = f(a)$ for every $a \in \mathbb{R}$ in this case.

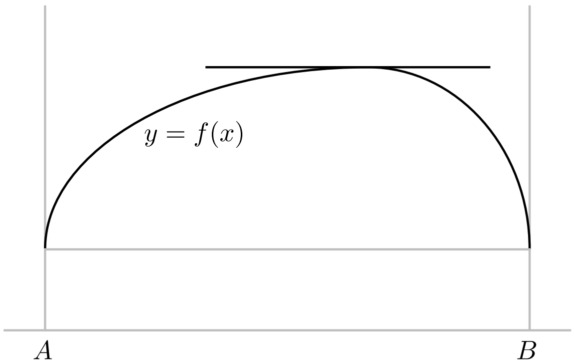
EXAMPLE 5.1.15. Consider the function $f(x) = \log x$. Then the inverse function is given by $g(y) = \mathrm{e}^y$. Then for every positive real number $a \in \mathbb{R}$, writing $b = \log a$, we have $f'(a)g'(b) = 1$ by Theorem 5E. It then follows from Example 5.1.14 that $f'(a)g(b) = 1$, and so

$$f'(a) = \frac{1}{g(b)} = \frac{1}{a}.$$

5.2. Some Important Results on Derivatives

In this section, we indicate some results which summarize, with rigour, the important role played by the derivative $f'(x)$ in the study of properties of a given function $f(x)$. The first of these results appears to be very restrictive, as it involves a hypothesis which is rarely satisfied.

THEOREM 5F. (ROLLE'S THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ exists for every $a \in (A, B)$. If $f(A) = f(B)$, then there exists $c \in (A, B)$ such that $f'(c) = 0$.*



PROOF. Since $f(x)$ is continuous in the closed interval $[A, B]$, it follows from Theorem 4H that there exist $x_1, x_2 \in [A, B]$ such that $f(x_1) \le f(x) \le f(x_2)$ for every $x \in [A, B]$.

Case 1. Suppose that both x_1 and x_2 are endpoints of the interval $[A, B]$. Since $f(A) = f(B)$, it follows that $f(x)$ is constant in the interval $[A, B]$, so that $f'(c) = 0$ for every $c \in (A, B)$.

Case 2. Suppose that $x_1 \in (A, B)$. Then $f(x)$ has a local minimum at $x = x_1$. We claim that $f'(x_1) = 0$. Suppose on the contrary that $f'(x_1) \ne 0$. Without loss of generality, assume that

$$f'(x_1) = \lim_{x\to x_1} \frac{f(x) - f(x_1)}{x - x_1} > 0.$$

Then there exists $\delta > 0$ such that

$$\left| \frac{f(x) - f(x_1)}{x - x_1} - f'(x_1) \right| < \frac{1}{2}|f'(x_1)| \qquad \text{whenever } 0 < |x - x_1| < \delta,$$

so that

$$\frac{f(x) - f(x_1)}{x - x_1} > 0 \qquad \text{whenever } 0 < |x - x_1| < \delta.$$

It follows that $f(x) - f(x_1) < 0$ if $x_1 - \delta < x < x_1$, contradicting that $f(x)$ has a local minimum at $x = x_1$.

Case 3. Suppose that $x_2 \in (A, B)$. Then $f(x)$ has a local maximum at $x = x_2$. A similar argument as in Case 2 gives $f'(x_2) = 0$. ◯

EXAMPLE 5.2.1. We can prove that between any two real roots of $\sin x = 0$ must lie a real root of $\cos x = 0$. To do this, let $f(x) = \sin x$, and let $A < B$ be any two real roots of $\sin x = 0$. Clearly $f(A) = f(B)$. Furthermore, all the other hypotheses of Rolle's theorem are satisfied. It follows that there exists $c \in (A, B)$ such that $f'(c) = 0$. Note, however, that $f'(x) = \cos x$.

EXAMPLE 5.2.2. Consider the polynomial $f(x) = x^3 + 3x^2 + 6x + 1$. We can prove that the polynomial equation $f(x) = 0$ has exactly one real root. Note that $f(-1) < 0$ and $f(1) > 0$. Applying the Intermediate value theorem to $f(x)$ in the closed interval $[-1, 1]$, we know that there exists $x_0 \in (-1, 1)$ such that $f(x_0) = 0$. It follows that the equation $f(x) = 0$ has at least one real root. Suppose that there are more than one real root. Let $A < B$ be two such roots. Then clearly $f(A) = f(B)$. Applying Rolle's theorem with $f(x) = x^3 + 3x^2 + 6x + 1$ in the interval $[A, B]$, we conclude that there exists $c \in (A, B)$ such that $f'(c) = 0$. Note, however, that $f'(x) = 3x^2 + 6x + 6 = 3(x^2 + 2x + 1 + 1) = 3(x + 1)^2 + 3 \neq 0$ for any $x \in \mathbb{R}$.

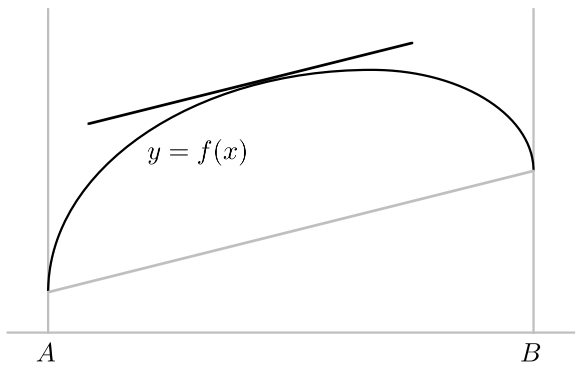
The hypotheses of Rolle's theorem are rather restrictive, in that we require the function to have equal values at the two end-points of the interval in question. However, this restriction is only deceptive, as we can use Rolle's theorem to establish the following more general result.

THEOREM 5G. (MEAN VALUE THEOREM) *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ exists for every $a \in (A, B)$. Then there exists $c \in (A, B)$ such that $f(B) - f(A) = f'(c)(B - A)$.*

To understand the Mean value theorem, it is easiest to rewrite the conclusion as

$$\frac{f(B) - f(A)}{B - A} = f'(c).$$

The left-hand side represents the slope of the line joining the points $(A, f(A))$ and $(B, f(B))$. It follows that the theorem merely says that the tangent to the curve is sometimes parallel to this line.



It is therefore clear that Rolle's theorem is a special case of the Mean value theorem. We now show that the Mean value theorem can be deduced fairly easily from Rolle's theorem.

Proof of Theorem 5G. Consider the function

$$g(x) = f(x) - \frac{f(B) - f(A)}{B - A}(x - A).$$

Then clearly $g(x)$ is continuous in the closed interval $[A, B]$, $g'(a)$ exists for every $a \in (A, B)$ and $g(A) = g(B)$. It follows from Rolle's theorem that there exists $c \in (A, B)$ such that $g'(c) = 0$. Note now that

$$g'(c) = f'(c) - \frac{f(B) - f(A)}{B - A}.$$

This completes the proof. ○

To illustrate the power of the Mean value theorem, we shall deduce the following simple but powerful consequences.

THEOREM 5H. *Suppose that a function $f(x)$ is continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ exists for every $a \in (A, B)$.*
(a) If $f'(a) = 0$ for every $a \in (A, B)$, then $f(x)$ is constant in $[A, B]$.
(b) If $f'(a) > 0$ for every $a \in (A, B)$, then $f(x)$ is strictly increasing in $[A, B]$.
(c) If $f'(a) < 0$ for every $a \in (A, B)$, then $f(x)$ is strictly decreasing in $[A, B]$.

Proof. Suppose that $A \le x_1 < x_2 \le B$. Applying the Mean value theorem to the function $f(x)$ in the closed interval $[x_1, x_2]$, we have

$$f(x_2) - f(x_1) = (x_2 - x_1)f'(c)$$

for some $c \in [x_1, x_2] \subseteq [A, B]$. It follows that

$$f(x_2) - f(x_1) = \begin{cases} = 0 & \text{in case (a)}, \\ > 0 & \text{in case (b)}, \\ < 0 & \text{in case (c)}, \end{cases}$$

giving the desired results. ○

We next discuss a generalization of the Mean value theorem to one involving two functions.

THEOREM 5J. (CAUCHY'S MEAN VALUE THEOREM) *Suppose that functions $f(x)$ and $g(x)$ are continuous in the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ with $A < B$. Suppose further that $f'(a)$ and $g'(a)$ exist for every $a \in (A, B)$, and that $g'(a)$ is non-zero for every $a \in (A, B)$. Then there exists $c \in (A, B)$ such that*

$$\frac{f(B) - f(A)}{g(B) - g(A)} = \frac{f'(c)}{g'(c)}.$$

Proof. We let $h(x) = f(x) - kg(x)$, where $k \in \mathbb{R}$ is a suitably chosen constant which ensures that $h(A) = h(B)$, so that

$$k = \frac{f(B) - f(A)}{g(B) - g(A)}.$$

Here we observe that the denominator $g(B) - g(A)$ is non-zero, in view of Rolle's theorem and the assumption that $g'(a)$ is non-zero for every $a \in (A, B)$. Clearly $h(x)$ is continuous in the closed interval

$[A, B]$, $h'(a)$ exists for every $a \in (A, B)$ and $h(A) = h(B)$. It follows from Rolle's theorem that there exists $c \in (A, B)$ such that $h'(c) = 0$. Note now that

$$\frac{h'(c)}{g'(c)} = \frac{f'(c)}{g'(c)} - k = \frac{f'(c)}{g'(c)} - \frac{f(B) - f(A)}{g(B) - g(A)}.$$

This completes the proof. ◯

We are now in a position to establish the following important result.

THEOREM 5K. (L'HÔPITAL'S RULE) *Suppose that functions $f(x)$ and $g(x)$ are differentiable in an open interval I containing the real number a. Suppose further that $f(a) = g(a) = 0$. Then*

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)},$$

provided that the limit on the right-hand side exists.

PROOF. For any $x \in I$ such that $x \neq a$, we apply Cauchy's mean value theorem to the closed interval $[a, x]$ if $x > a$ and to the closed interval $[x, a]$ if $x < a$. It is easy to check that the hypotheses of Cauchy's mean value theorem are satisfied. Hence there exists $c \in (a, x)$ or $c \in (x, a)$ such that

$$\frac{f(x)}{g(x)} = \frac{f(x) - f(a)}{g(x) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Clearly $c \to a$ as $x \to a$. Hence

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{c \to a} \frac{f'(c)}{g'(c)},$$

and the result follows. ◯

5.3. Stationary Points and Second Derivatives

DEFINITIONS.

(1) A function $f(x)$ is said to have a local maximum at $x = a$ if there is an open interval I containing the real number a and such that $f(x) \le f(a)$ for every $x \in I$.
(2) A function $f(x)$ is said to have a local minimum at $x = a$ if there is an open interval I containing the real number a and such that $f(x) \ge f(a)$ for every $x \in I$.
(3) A function $f(x)$ is said to have a stationary point at $x = a$ if $f'(a) = 0$.

EXAMPLE 5.3.1. Consider the function $f(x) = x^2$. Since $f'(x) = 2x$ for every $x \in \mathbb{R}$, the only stationary point is at $x = 0$. On the other hand, note that for every $x \neq 0$, we have $f(x) = x^2 > 0 = f(0)$. It follows that there is a local minimum at $x = 0$.

EXAMPLE 5.3.2. Consider the function $f(x) = x^3$. Since $f'(x) = 3x^2$ for every $x \in \mathbb{R}$, the only stationary point is at $x = 0$. On the other hand, note that for every $x < 0$, we have $f(x) = x^3 < 0 = f(0)$, whereas for every $x > 0$, we have $f(x) = x^3 > 0 = f(0)$. It follows that $x = 0$ does not represent a local minimum or a local maximum.

To detect a local maximum or local minimum, we have the following result.

THEOREM 5L. *Suppose that I is an open interval containing a. Suppose further that a function $f(x)$ is continuous in I, and differentiable at every $x \in I$, except possibly at $x = a$.*
(a) If $f'(x) > 0$ for every $x < a$ in I and $f'(x) < 0$ for every $x > a$ in I, then the function $f(x)$ has a local maximum at $x = a$.
(b) If $f'(x) < 0$ for every $x < a$ in I and $f'(x) > 0$ for every $x > a$ in I, then the function $f(x)$ has a local minimum at $x = a$.

PROOF. Suppose that $x \in I$ and $x \neq a$. By the Mean value theorem, there exists a real number c in the open interval with endpoints a and x such that $f(x) - f(a) = (x - a)f'(c)$.

(a) Since $f'(c) > 0$ if $x < a$ and $f'(c) < 0$ if $x > a$, we clearly have $f(x) - f(a) < 0$. Hence $f(x)$ has a local maximum at $x = a$.

(b) Since $f'(c) < 0$ if $x < a$ and $f'(c) > 0$ if $x > a$, we clearly have $f(x) - f(a) > 0$. Hence $f(x)$ has a local minimum at $x = a$. ○

EXAMPLE 5.3.3. Consider the function $f(x) = 2x^3 - 9x^2 + 12x - 5$. Since

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x - 1)(x - 2)$$

for every $x \in \mathbb{R}$, it is clear that the only stationary points are at $x = 1$ and $x = 2$. To determine whether either of these represents a local maximum or a local minimum, we study the function $f'(x)$ more closely. It is easy to see that

$$f'(x) \begin{cases} > 0 & \text{if } x \in (0, 1), \\ < 0 & \text{if } x \in (1, 2), \\ > 0 & \text{if } x \in (2, 3). \end{cases}$$

It follows that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 2$.

If the first derivative measures the rate of change of a function, then the second derivative measures the rate of change of the first derivative. Since the first derivative represents the slope of the tangent to the curve, it follows that the second derivative measures the rate of change of this slope. The following result is suggested by heuristics bases on these ideas.

THEOREM 5M. *Suppose that I is an open interval containing a real number a. Suppose further that the function $f(x)$ is differentiable at every $x \in I$, and that $f'(a) = 0$.*
(a) If $f''(a) < 0$, then the function $f(x)$ has a local maximum at $x = a$.
(b) If $f''(a) > 0$, then the function $f(x)$ has a local minimum at $x = a$.

PROOF. We shall only prove (a), as the proof for (b) is similar. Since

$$f''(a) = \lim_{x \to a} \frac{f'(x) - f'(a)}{x - a} < 0,$$

it follows that there exists $\delta > 0$ such that

$$\left| \frac{f'(x) - f'(a)}{x - a} - f''(a) \right| < \frac{1}{2}|f''(a)| \qquad \text{whenever } 0 < |x - a| < \delta,$$

so that

$$\frac{f'(x) - f'(a)}{x - a} < 0 \qquad \text{whenever } 0 < |x - a| < \delta.$$

Now let $I = (a - \delta, a + \delta)$. Then it is easy to see that $f'(x) > 0$ for every $x < a$ in I and $f'(x) < 0$ for every $x > a$ in I. It now follows from Theorem 5L that $f(x)$ has a local maximum at $x = a$. ○

Example 5.3.4. Consider the function $f(x) = 2x^3 - 9x^2 + 12x - 5$, as discussed earlier in Example 5.3.3. Since

$$f'(x) = 6x^2 - 18x + 12 = 6(x^2 - 3x + 2) = 6(x-1)(x-2)$$

for every $x \in \mathbb{R}$, it is clear that the only stationary points are at $x = 1$ and $x = 2$. On the other hand, we have $f''(x) = 12x - 18$ for every $x \in \mathbb{R}$, so that $f''(1) < 0$ and $f''(2) > 0$. It follows that $f(x)$ has a local maximum at $x = 1$ and a local minimum at $x = 2$.

5.4. Series Expansion

The purpose of this section is to show that if a given function has derivatives of all orders, then it has a nice power series expansion. We begin by establishing the following generalized version of the Mean value theorem.

THEOREM 5N. (TAYLOR'S THEOREM) *Suppose that $n \in \mathbb{N}$. Suppose further that a function $f(x)$ satisfies the following conditions:*
(a) $f(x)$ and its first $(n-1)$ derivatives $f'(x), f''(x), \ldots, f^{(n-1)}(x)$ are continuous in the closed interval $[a, a+h]$; and
(b) the n-th derivative exists in the open interval $(a, a+h)$.
Then

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{h^n}{n!}f^{(n)}(a+\theta h),$$

where $\theta \in \mathbb{R}$ satisfies $0 < \theta < 1$.

Remark. Taylor's theorem is sometimes known as the Mean value theorem of the n-th order. Note that for $n = 1$, Taylor's theorem reduces to the Mean value theorem.

Proof of Theorem 5N. For every $t \in [0, h]$, write

$$g(t) = f(a+t) - f(a) - tf'(a) - \ldots - \frac{t^{n-1}}{(n-1)!}f^{(n-1)}(a) - \frac{t^n}{n!}C, \tag{1}$$

where we shall choose C to ensure that $g(h) = 0$. It is easy to check that

$$g(0) = g'(0) = \ldots = g^{(n-1)}(0) = 0.$$

We now proceed to use Rolle's theorem n times. Since $g(0) = g(h) = 0$, there exists $h_1 \in (0, h)$ such that $g'(h_1) = 0$. Since $g'(0) = g'(h_1) = 0$, there exists $h_2 \in (0, h_1)$ such that $g''(h_2) = 0$, and so on. Finally, since $g^{(n-1)}(0) = g^{(n-1)}(h_{n-1}) = 0$, there exists $h_n \in (0, h_{n-1})$ such that $g^{(n)}(h_n) = 0$. Clearly $0 < h_n < h$, and so $h_n = \theta h$ for some $\theta \in \mathbb{R}$ satisfying $0 < \theta < 1$. Observe now that

$$g^{(n)}(t) = f^{(n)}(a+t) - C.$$

It follows that $C = f^{(n)}(a+\theta h)$. The result follows on substituting this into (1), letting $t = h$ and noting that $g(h) = 0$. ○

In Taylor's theorem, we can write

$$f(a+h) = S_n + R_n,$$

where

$$S_n = f(a) + hf'(a) + \frac{h^2}{2!}f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!}f^{(n-1)}(a)$$

and

$$R_n = \frac{h^n}{n!}f^{(n)}(a + \theta h). \tag{2}$$

If $R_n \to 0$ as $n \to \infty$, then $S_n \to f(a+h)$ as $n \to \infty$. We therefore have the following series version of Taylor's theorem.

THEOREM 5P. (TAYLOR SERIES) *Suppose that a function $f(x)$ satisfies the following conditions:*
(a) $f(x)$ and all its derivatives $f'(x), f''(x), \ldots$ are continuous in the closed interval $[a, a+h]$; and
(b) the sequence R_n defined by (2) converges to 0 as $n \to \infty$.
Then

$$f(a+h) = \sum_{n=0}^{\infty} \frac{h^n}{n!} f^{(n)}(a),$$

with the convention that $0! = 1$.

REMARK. The Maclaurin series is the Taylor series in the special case $a = 0$. Under suitable conditions, we have

$$f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!} f^{(n)}(0). \tag{3}$$

EXAMPLE 5.4.1. Consider the function $f(x) = \mathrm{e}^x$. Then $f(x)$ has derivatives of all order, all equal to e^x. Note that $f^{(n)}(0) = 1$ for every $n \in \mathbb{N} \cup \{0\}$. It follows that the Maclaurin series of the exponential function is given by

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

This is the exponential series.

EXAMPLE 5.4.2. Consider the function $f(x) = \log(1+x)$. Then $f(x)$ has derivatives of all order near $x = 0$. Furthermore, it can be proved by induction that for every $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = \frac{(-1)^{n-1}(n-1)!}{(1+x)^n},$$

so that $f^{(n)}(0) = (-1)^{n-1}(n-1)!$. Note also that $f(0) = 0$. It follows that the Maclaurin series for the function is given by

$$\log(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}.$$

This is the logarithmic series.

EXAMPLE 5.4.3. Consider the function $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R} \setminus \{0, 1, 2, 3, \ldots\}$. Then $f(x)$ has derivatives of all order near $x = 0$. Furthermore, for every $n \in \mathbb{N}$, we have

$$f^{(n)}(x) = \alpha(\alpha-1)\ldots(\alpha-n+1)(1+x)^{\alpha-n},$$

so that

$$f^{(n)}(0) = \alpha(\alpha - 1)\dots(\alpha - n + 1).$$

Note also that $f(0) = 1$. It follows that the Maclaurin series for the function is given by

$$(1 + x)^\alpha = \sum_{n=1}^{\infty} \frac{\alpha(\alpha - 1)\dots(\alpha - n + 1)}{n!} x^n.$$

This is the Extended binomial theorem.

EXAMPLE 5.4.4. Consider the function $f(x) = (1 + x)^n$, where $n \in \mathbb{N}$. Then $f(x)$ has derivatives of all order near $x = 0$. Furthermore, for every $r = 1, \dots, n$, we have

$$f^{(r)}(x) = n(n - 1)\dots(n - r + 1)(1 + x)^{n-r},$$

so that

$$f^{(r)}(0) = n(n - 1)\dots(n - r + 1).$$

On the other hand, for every natural number $r > n$, we have $f^{(r)}(x) = 0$. Note also that $f(0) = 1$. It follows that the Maclaurin series for the function has zero coefficients beyond the term x^n and is given by

$$(1 + x)^n = \sum_{r=0}^{n} \frac{n(n - 1)\dots(n - r + 1)}{r!} x^r.$$

This is a special case of the Binomial theorem.

Problems for Chapter 5

1. a) Suppose that $f(x)$ and $g(x)$ are twice differentiable at $x = a$. Show that

$$(fg)''(a) = f''(a)g(a) + 2f'(a)g'(a) + f(a)g''(a).$$

b) Suppose that $f(x)$ and $g(x)$ are three times differentiable at $x = a$. Obtain a corresponding formula for $(fg)'''(a)$.
c) Suppose that $f(x)$ and $g(x)$ are n times differentiable at $x = a$. Analyze the results in parts (a) and (b), make a guess for the corresponding formula for $(fg)^{(n)}(a)$, and prove your formula by induction on n.

2. Suppose that $f''(a)$ exists. Prove that

$$\lim_{h\to 0} \frac{f(a+h) - 2f(a) + f(a-h)}{h^2} = f''(a).$$

3. Let $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
a) Show that $f(x)$ is continuous at $x = 0$.
b) Find the derivative of $f(x)$ when $x \neq 0$.
c) Show that $f(x)$ is not differentiable at $x = 0$.

4. Let $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$
a) Prove that $f'(x)$ exists for every real number x.
b) Find $f'(0)$.
c) Find $f'(x)$ when $x \neq 0$.
d) Prove that $f'(x)$ is not continuous at $x = 0$.

5. Construct a function $g(x)$ for which $g'(0) > 0$, but there is no interval $(-A, A)$ in which $g(x)$ is a strictly increasing function.
[Hint: Try $g(x) = f(x) + kx$, where k is a suitable constant and $f(x)$ is given in Problem 4.]

6. Consider the function $f(x) = |x| - 3$.
a) Show that $f(x)$ is differentiable at $x = a$ for every non-zero $a \in \mathbb{R}$.
b) Comment in view of Theorem 5L.

7. Suppose that the function $f(x)$ satisfies $f(0) = 0$, $f'(0) = 0$ and $f''(0) > 0$.
a) Explain why there exists $\delta > 0$ such that $\dfrac{f'(x) - f'(0)}{x - 0} > 0$ for every non-zero $x \in (-\delta, \delta)$.
b) Deduce that $f'(x) > 0$ for every $x \in (0, \delta)$, and that $f'(x) < 0$ for every $x \in (-\delta, 0)$.
c) Use Rolle's theorem to show that $f(x) \neq 0$ for every non-zero $x \in (-\delta, \delta)$.
d) Use the Mean value theorem to show that $f(x) > 0$ for every non-zero $x \in (-\delta, \delta)$.

8. Consider the function $f(x) = x^{2/3}$ in the closed interval $[-1, 1]$.
a) Show that $f(-1) = f(1)$.
b) Show that there is no number $c \in (-1, 1)$ such that $f'(c) = 0$.
c) Show that $f(x)$ is not differentiable at $x = 0$.
d) Explain why the conclusion of Rolle's theorem does not hold.

9. Explain why $x = 1$ is the only real solution of the equation $x^3 - 3x^2 + 9x - 7 = 0$.

10. Use the relevant theorems to prove that the equation $\mathrm{e}^x = 3 - x$ has exactly one real solution.

11. Show that the equation $3x - 2 + \cos\dfrac{\pi x}{2} = 0$ has exactly one real root.

12. Use the Mean Value Theorem to prove the inequality $|\sin A - \sin B| \le |A - B|$ for all real numbers A and B.

13. Let $f(x) = \tan x - x$. Find $f(0)$ and use the derivative $f'(x)$ to prove that $\tan x > x$ for every x satisfying $0 < x < \pi/2$.

14. Suppose that $p(x)$ is a polynomial, and that $k \in \mathbb{R}$ is a constant. Suppose further that $A < B$ are consecutive roots of the equation $p(x) = 0$.
    a) Write $p(x) = (x - A)^m (x - B)^n q(x)$, where $q(A) \neq 0$ and $q(B) \neq 0$. Prove that if we write $p'(x) = (x - A)^{m-1}(x - B)^{n-1} r(x)$, then $r(A)$ and $r(B)$ have opposite signs.
    b) Hence, or otherwise, prove that there is a root of the equation $p'(x) + kp(x) = 0$ in the interval $[A, B]$.

15. Suppose that a function $f(x)$ is differentiable at every $x \in [A, B]$. Prove that $f'(x)$ takes every value between $f'(A)$ and $f'(B)$.

16. Use L'Hopital's rule to find each of the following:
    a) $\displaystyle\lim_{x\to 0} \frac{x - \sin x}{x^3}$ b) $\displaystyle\lim_{x\to 0^+} x^{2x}$ c) $\displaystyle\lim_{x\to 0} \frac{\tan x - x}{x^3}$

17. Find the Maclaurin expansion of the functions $\sin x$ and $\cos x$.

18. Find all the terms up to and including x^3 in the Taylor expansion of each of the following functions:
    a) $f(x) = (x + 1)\sin x$ b) $f(x) = \mathrm{e}^x \cos x$ c) $f(x) = \tan x$