FUNDAMENTALS OF ANALYSIS

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Chapter 6

THE RIEMANN INTEGRAL

6.1. Introduction

Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Suppose further that

$$\Delta: \ A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

is a dissection of the interval $[A,B]$.

DEFINITION. The sums

$$s(f,\Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x\in[x_{i-1},x_i]} f(x) \qquad \text{and} \qquad S(f,\Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x\in[x_{i-1},x_i]} f(x)$$

are called respectively the lower Riemann sum and the upper Riemann sum of $f(x)$ corresponding to the dissection Δ.

EXAMPLE 6.1.1. Consider the function $f(x) = x^2$ in the interval $[0,1]$. Suppose that $n \in \mathbb{N}$ is given and fixed. Let us consider a dissection

$$\Delta_n: \ 0 = x_0 < x_1 < x_2 < \ldots < x_n = 1$$

of the interval $[0,1]$, where $x_i = i/n$ for every $i = 0,1,2,\ldots,n$. For every $i = 1,2,\ldots,n$, we have

$$\inf_{x\in[x_{i-1},x_i]} f(x) = \inf_{\frac{i-1}{n} \le x \le \frac{i}{n}} x^2 = \frac{(i-1)^2}{n^2} \qquad \text{and} \qquad \sup_{x\in[x_{i-1},x_i]} f(x) = \sup_{\frac{i-1}{n} \le x \le \frac{i}{n}} x^2 = \frac{i^2}{n^2}.$$

It follows that

$$s(f,\Delta_n)=\sum_{i=1}^{n}(x_i-x_{i-1})\inf_{x\in[x_{i-1},x_i]}f(x)=\sum_{i=1}^{n}\frac{(i-1)^2}{n^3}=\frac{(n-1)n(2n-1)}{6n^3}$$

and

$$S(f,\Delta_n)=\sum_{i=1}^{n}(x_i-x_{i-1})\sup_{x\in[x_{i-1},x_i]}f(x)=\sum_{i=1}^{n}\frac{i^2}{n^3}=\frac{n(n+1)(2n+1)}{6n^3}.$$

Note that $s(f,\Delta_n)\le S(f,\Delta_n)$, and that both terms converge to $\frac{1}{3}$ as $n\to\infty$.

THEOREM 6A. *Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Suppose further that Δ' and Δ are dissections of the interval $[A,B]$, and that $\Delta'\subseteq\Delta$. Then*

$$s(f,\Delta')\le s(f,\Delta)\qquad\textit{and}\qquad S(f,\Delta)\le S(f,\Delta').$$

PROOF. Suppose that $x'<x''$ are consecutive dissection points of Δ', and suppose that

$$x'=y_0<y_1<\ldots<y_m=x''$$

are all the dissection points of Δ in the interval $[x',x'']$. Then, drawing a picture if necessary, it is easy to see that

$$\sum_{i=1}^{m}(y_i-y_{i-1})\inf_{x\in[y_{i-1},y_i]}f(x)\ge\sum_{i=1}^{m}(y_i-y_{i-1})\inf_{x\in[x',x'']}f(x)=(x''-x')\inf_{x\in[x',x'']}f(x)$$

and

$$\sum_{i=1}^{m}(y_i-y_{i-1})\sup_{x\in[y_{i-1},y_i]}f(x)\le\sum_{i=1}^{m}(y_i-y_{i-1})\sup_{x\in[x',x'']}f(x)=(x''-x')\sup_{x\in[x',x'']}f(x).$$

The result follows on summing over all consecutive points of the dissection Δ'. ○

THEOREM 6B. *Suppose that a function $f(x)$ is bounded on the interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Suppose further that Δ' and Δ'' are dissections of the interval $[A,B]$. Then*

$$s(f,\Delta')\le S(f,\Delta'').$$

PROOF. Consider the dissection $\Delta=\Delta'\cup\Delta''$ of $[A,B]$. Then it follows from Theorem 6A that

$$s(f,\Delta')\le s(f,\Delta)\qquad\text{and}\qquad S(f,\Delta)\le S(f,\Delta''). \tag{1}$$

On the other hand, it is easy to check that

$$s(f,\Delta)\le S(f,\Delta). \tag{2}$$

The result follows on combining (1) and (2). ○

DEFINITION. The real numbers

$$I^-(f,A,B)=\sup_{\Delta}s(f,\Delta)\qquad\text{and}\qquad I^+(f,A,B)=\inf_{\Delta}S(f,\Delta),$$

where the supremum and infimum are taken over all dissections Δ of $[A,B]$, are called respectively the lower integral and the upper integral of $f(x)$ over $[A,B]$.

Remark. Since $f(x)$ is bounded on $[A, B]$, it follows that $s(f, \Delta)$ and $S(f, \Delta)$ are bounded above and below. This guarantees the existence of $I^-(f, A, B)$ and $I^+(f, A, B)$.

THEOREM 6C. *Suppose that a function $f(x)$ is bounded on the interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Then $I^-(f, A, B) \le I^+(f, A, B)$.*

Proof. Suppose that Δ' is a dissection of $[A, B]$. Then it follows from Theorem 6B that

$$s(f, \Delta') \le S(f, \Delta)$$

for every dissection Δ of $[A, B]$. Keeping Δ' fixed and taking the infimum over all dissections Δ of $[A, B]$, we conclude that

$$s(f, \Delta') \le \inf_{\Delta} S(f, \Delta) = I^+(f, A, B).$$

Taking now the supremum over all dissections Δ' of $[A, B]$, we conclude that

$$I^+(f, A, B) \ge \sup_{\Delta'} s(f, \Delta') = I^-(f, A, B).$$

The result follows. ○

Definition. Suppose that $I^-(f, A, B) = I^+(f, A, B)$. Then we say that the function $f(x)$ is Riemann integrable over $[A, B]$, denoted by $f \in \mathcal{R}([A, B])$, and write

$$\int_A^B f(x)\,\mathrm{d}x = I^-(f, A, B) = I^+(f, A, B).$$

Example 6.1.2. Let us return to Example 6.1.1, and consider again the function $f(x) = x^2$ in the interval $[0, 1]$. Recall that both $s(f, \Delta_n)$ and $S(f, \Delta_n)$ converge to $\frac{1}{3}$ as $n \to \infty$. It follows that

$$I^-(f, 0, 1) \ge \frac{1}{3} \qquad \text{and} \qquad I^+(f, 0, 1) \le \frac{1}{3}.$$

In view of Theorem 6C, we must have

$$I^-(f, 0, 1) = I^+(f, 0, 1) = \frac{1}{3},$$

so that

$$\int_0^1 x^2\,\mathrm{d}x = \frac{1}{3}.$$

We can establish the following characterization of Riemann integrable functions in terms of Riemann sums.

THEOREM 6D. *Suppose that a function $f(x)$ is bounded on the interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$. Then the following two statements are equivalent:*
(a) $f \in \mathcal{R}([A, B])$.
(b) Given any $\epsilon > 0$, there exists a dissection Δ of $[A, B]$ such that

$$S(f, \Delta) - s(f, \Delta) < \epsilon. \tag{3}$$

PROOF. ((a)⇒(b)) If $f \in \mathcal{R}([A,B])$, then

$$\sup_{\Delta} s(f,\Delta) = \inf_{\Delta} S(f,\Delta), \tag{4}$$

where the supremum and infimum are taken over all dissections Δ of $[A,B]$. For every $\epsilon > 0$, there exist dissections Δ_1 and Δ_2 of $[A,B]$ such that

$$s(f,\Delta_1) > \sup_{\Delta} s(f,\Delta) - \frac{\epsilon}{2} \qquad \text{and} \qquad S(f,\Delta_2) < \inf_{\Delta} S(f,\Delta) + \frac{\epsilon}{2}. \tag{5}$$

Let $\Delta = \Delta_1 \cup \Delta_2$. Then by Theorem 6A, we have

$$s(f,\Delta) \ge s(f,\Delta_1) \qquad \text{and} \qquad S(f,\Delta) \le S(f,\Delta_2). \tag{6}$$

The inequality (3) now follows on combining (4)–(6).

((b)⇒(a)) Suppose that $\epsilon > 0$ is given. We can choose a dissection Δ of $[A,B]$ such that (3) holds. Clearly

$$s(f,\Delta) \le I^-(f,A,B) \le I^+(f,A,B) \le S(f,\Delta). \tag{7}$$

Combining (3) and (7), we conclude that $0 \le I^+(f,A,B) - I^-(f,A,B) < \epsilon$. Note now that $\epsilon > 0$ is arbitrary, and that $I^+(f,A,B) - I^-(f,A,B)$ is independent of ϵ. It follows that we must have $I^+(f,A,B) - I^-(f,A,B) = 0$. ○

6.2. Properties of the Riemann Integral

In this section, we shall study some simple but useful properties of the Riemann integral. We begin by studying the arithmetic of Riemann integrals.

THEOREM 6E. *Suppose that $f,g \in \mathcal{R}([A,B])$, where $A,B \in \mathbb{R}$ and $A < B$. Then the following statements hold:*

(a) We have $f+g \in \mathcal{R}([A,B])$, and $\displaystyle\int_A^B (f(x)+g(x))\,\mathrm{d}x = \int_A^B f(x)\,\mathrm{d}x + \int_A^B g(x)\,\mathrm{d}x.$

(b) For every $c \in \mathbb{R}$, we have $cf \in \mathcal{R}([A,B])$, and $\displaystyle\int_A^B cf(x)\,\mathrm{d}x = c\int_A^B f(x)\,\mathrm{d}x.$

(c) If $f(x) \ge 0$ for every $x \in [A,B]$, then $\displaystyle\int_A^B f(x)\,\mathrm{d}x \ge 0.$

(d) If $f(x) \le g(x)$ for every $x \in [A,B]$, then $\displaystyle\int_A^B f(x)\,\mathrm{d}x \le \int_A^B g(x)\,\mathrm{d}x.$

PROOF. (a) Since $f,g \in \mathcal{R}([A,B])$, it follows from Theorem 6D that for every $\epsilon > 0$, there exist dissections Δ_1 and Δ_2 of $[A,B]$ such that

$$S(f,\Delta_1) - s(f,\Delta_1) < \frac{\epsilon}{2} \qquad \text{and} \qquad S(g,\Delta_2) - s(g,\Delta_2) < \frac{\epsilon}{2}.$$

Let $\Delta = \Delta_1 \cup \Delta_2$. Then in view of Theorem 6A, we have

$$S(f,\Delta) - s(f,\Delta) < \frac{\epsilon}{2} \qquad \text{and} \qquad S(g,\Delta) - s(g,\Delta) < \frac{\epsilon}{2}. \tag{8}$$

Suppose that the dissection Δ is given by $\Delta:\ A = x_0 < x_1 < x_2 < \ldots < x_n = B$. It is easy to see that for every $i = 1,\ldots,n$, we have

$$\sup_{x\in[x_{i-1},x_i]} (f(x)+g(x)) \le \sup_{x\in[x_{i-1},x_i]} f(x) + \sup_{x\in[x_{i-1},x_i]} g(x)$$

and

$$\inf_{x\in[x_{i-1},x_i]}(f(x)+g(x)) \geq \inf_{x\in[x_{i-1},x_i]} f(x) + \inf_{x\in[x_{i-1},x_i]} g(x).$$

It follows that

$$S(f+g,\Delta) \leq S(f,\Delta)+S(g,\Delta) \qquad \text{and} \qquad s(f+g,\Delta) \geq s(f,\Delta)+s(g,\Delta). \tag{9}$$

Combining (8) and (9), we have

$$S(f+g,\Delta)-s(f+g,\Delta) \leq (S(f,\Delta)-s(f,\Delta))+(S(g,\Delta)-s(g,\Delta)) < \epsilon.$$

It now follows from Theorem 6D that $f+g \in \mathcal{R}([A,B])$. To establish the second assertion, suppose now that Δ_1 and Δ_2 are any two dissections of $[A,B]$. As before, let $\Delta=\Delta_1\cup\Delta_2$. Then in view of Theorem 6A and (9), we have

$$S(f,\Delta_1)+S(g,\Delta_2) \geq S(f,\Delta)+S(g,\Delta) \geq S(f+g,\Delta) \geq I^+(f+g,A,B),$$

so that

$$S(g,\Delta_2) \geq I^+(f+g,A,B)-S(f,\Delta_1).$$

Keeping Δ_1 fixed and taking the infimum over all dissections Δ_2 of $[A,B]$, we have

$$I^+(g,A,B) \geq I^+(f+g,A,B)-S(f,\Delta_1),$$

so that

$$S(f,\Delta_1) \geq I^+(f+g,A,B)-I^+(g,A,B).$$

Taking the infimum over all dissections Δ_1 of $[A,B]$, we have

$$I^+(f,A,B) \geq I^+(f+g,A,B)-I^+(g,A,B),$$

so that

$$I^+(f+g,A,B) \leq I^+(f,A,B)+I^+(g,A,B). \tag{10}$$

Similarly, in view of Theorem 6A and (9), we have

$$s(f,\Delta_1)+s(g,\Delta_2) \leq s(f,\Delta)+s(g,\Delta) \leq s(f+g,\Delta) \leq I^-(f+g,A,B),$$

so that

$$s(g,\Delta_2) \leq I^-(f+g,A,B)-s(f,\Delta_1).$$

Keeping Δ_1 fixed and taking the supremum over all dissections Δ_2 of $[A,B]$, we have

$$I^-(g,A,B) \leq I^-(f+g,A,B)-s(f,\Delta_1),$$

so that

$$s(f,\Delta_1) \leq I^-(f+g,A,B)-I^-(g,A,B).$$

Taking the supremum over all dissections Δ_1 of $[A,B]$, we have

$$I^-(f,A,B) \leq I^-(f+g,A,B)-I^-(g,A,B),$$

so that

$$I^-(f,A,B) + I^-(g,A,B) \le I^-(f+g,A,B). \tag{11}$$

Combining (10) and (11), we have

$$I^-(f,A,B) + I^-(g,A,B) \le I^-(f+g,A,B) = I^+(f+g,A,B) \le I^+(f,A,B) + I^+(g,A,B). \tag{12}$$

Clearly $I^-(f,A,B) = I^+(f,A,B)$ and $I^-(g,A,B) = I^+(g,A,B)$, and so equality must hold everywhere in (12). In particular, we have $I^+(f,A,B) + I^+(g,A,B) = I^+(f+g,A,B)$.

(b) The case $c = 0$ is trivial. Suppose now that $c > 0$. Since $f \in \mathcal{R}([A,B])$, it follows from Theorem 6D that for every $\epsilon > 0$, there exists a dissection Δ of $[A,B]$ such that

$$S(f,\Delta) - s(f,\Delta) < \frac{\epsilon}{c}.$$

It is easy to see that

$$S(cf,\Delta) = cS(f,\Delta) \qquad \text{and} \qquad s(cf,\Delta) = cs(f,\Delta). \tag{13}$$

Hence

$$S(cf,\Delta) - s(cf,\Delta) < \epsilon.$$

It follows from Theorem 6D that $cf \in \mathcal{R}([A,B])$. Also, (13) clearly implies $I^+(cf,A,B) = cI^+(f,A,B)$. Suppose next that $c < 0$. Since $f \in \mathcal{R}([A,B])$, it follows from Theorem 6D that for every $\epsilon > 0$, there exists a dissection Δ of $[A,B]$ such that

$$S(f,\Delta) - s(f,\Delta) < -\frac{\epsilon}{c}.$$

It is easy to see that

$$S(cf,\Delta) = cs(f,\Delta) \qquad \text{and} \qquad s(cf,\Delta) = cS(f,\Delta). \tag{14}$$

Hence

$$S(cf,\Delta) - s(cf,\Delta) < \epsilon.$$

It follows from Theorem 6D that $cf \in \mathcal{R}([A,B])$. Also, (14) clearly implies $I^+(cf,A,B) = cI^-(f,A,B)$.

(c) Note simply that

$$\int_A^B f(x)\,\mathrm{d}x \ge (B-A) \inf_{x\in[A,B]} f(x),$$

where the right hand side is the lower sum corresponding to the trivial dissection.

(d) Note that $g - f \in \mathcal{R}([A,B])$ in view of (a) and (b). We apply part (c) to the function $g - f$. ○

Next, we investigate the question of breaking up the interval $[A,B]$ of integration.

THEOREM 6F. *Suppose that $f \in \mathcal{R}([A,B])$, where $A, B \in \mathbb{R}$ and $A < B$. Then for every real number $C \in (A,B)$, we have $f \in \mathcal{R}([A,C])$ and $f \in \mathcal{R}([C,B])$. Furthermore, we have*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^C f(x)\,\mathrm{d}x + \int_C^B f(x)\,\mathrm{d}x. \tag{15}$$

PROOF. We shall first show that for every $C', C'' \in \mathbb{R}$ satisfying $A \le C' < C'' \le B$, we have $f \in \mathcal{R}([C', C''])$. Since $f \in \mathcal{R}([A, B])$, it follows from Theorem 6D that given any $\epsilon > 0$, there exists a dissection Δ^* of $[A, B]$ such that

$$S(f, \Delta^*) - s(f, \Delta^*) < \epsilon.$$

It follows from Theorem 6A that the dissection $\Delta = \Delta^* \cup \{C', C''\}$ of $[A, B]$ satisfies

$$S(f, \Delta) - s(f, \Delta) < \epsilon. \tag{16}$$

Suppose that the dissection Δ is given by $\Delta : \; A = x_0 < x_1 < x_2 < \ldots < x_n = B$. Then there exist $k', k'' \in \{0, 1, 2, \ldots, n\}$ satisfying $k' < k''$ such that $C' = x_{k'}$ and $C'' = x_{k''}$. It follows that

$$\Delta_0 : \; C' = x_{k'} < x_{k'+1} < x_{k'+2} < \ldots < x_{k''} = C''$$

is a dissection of $[C', C'']$. Furthermore,

$$\begin{aligned} S(f, \Delta_0) - s(f, \Delta_0) &= \sum_{i=k'+1}^{k''} (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) \\ &\le \sum_{i=1}^{n} (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) \\ &= S(f, \Delta) - s(f, \Delta) < \epsilon, \end{aligned}$$

in view of (16). It now follows from Theorem 6D that $f \in \mathcal{R}([C', C''])$. To establish (15), note that by definition, we have

$$\int_A^B f(x) \, dx = \inf_{\Delta} S(f, \Delta), \tag{17}$$

while

$$\int_A^C f(x) \, dx = \inf_{\Delta_1} S(f, \Delta_1) \qquad \text{and} \qquad \int_C^B f(x) \, dx = \inf_{\Delta_2} S(f, \Delta_2). \tag{18}$$

Here Δ, Δ_1 and Δ_2 run over all dissections of $[A, B]$, $[A, C]$ and $[C, B]$ respectively. The identity (15) will follow from (17) and (18) if we can show that

$$\inf_{\Delta} S(f, \Delta) = \inf_{\Delta_1} S(f, \Delta_1) + \inf_{\Delta_2} S(f, \Delta_2). \tag{19}$$

Suppose first of all that Δ is a dissection of $[A, B]$. Then we can write $\Delta \cup \{C\} = \Delta' \cup \Delta''$, where Δ' and Δ'' are dissections of $[A, C]$ and $[C, B]$ respectively. By Theorem 6A, we have

$$S(f, \Delta) \ge S(f, \Delta \cup \{C\}) = S(f, \Delta') + S(f, \Delta'').$$

Clearly

$$S(f, \Delta') + S(f, \Delta'') \ge \inf_{\Delta_1} S(f, \Delta_1) + \inf_{\Delta_2} S(f, \Delta_2).$$

Hence

$$S(f, \Delta) \ge \inf_{\Delta_1} S(f, \Delta_1) + \inf_{\Delta_2} S(f, \Delta_2).$$

Taking the infimum over all dissections Δ of $[A, B]$, we conclude that

$$\inf_{\Delta} S(f, \Delta) \ge \inf_{\Delta_1} S(f, \Delta_1) + \inf_{\Delta_2} S(f, \Delta_2). \tag{20}$$

To establish the opposite inequality, suppose next that Δ_1 and Δ_2 are dissections of $[A, C]$ and $[C, B]$ respectively. Then $\Delta_1 \cup \Delta_2$ is a dissection of $[A, B]$, and

$$S(f, \Delta_1) + S(f, \Delta_2) = S(f, \Delta_1 \cup \Delta_2) \geq \inf_{\Delta} S(f, \Delta).$$

This implies that

$$S(f, \Delta_1) \geq \inf_{\Delta} S(f, \Delta) - S(f, \Delta_2).$$

Keeping Δ_2 fixed and taking the infimum over all dissections Δ_1 of $[A, C]$, we have

$$\inf_{\Delta_1} S(f, \Delta_1) \geq \inf_{\Delta} S(f, \Delta) - S(f, \Delta_2),$$

and so

$$S(f, \Delta_2) \geq \inf_{\Delta} S(f, \Delta) - \inf_{\Delta_1} S(f, \Delta_1).$$

Taking the infimum over all dissections Δ_2 of $[C, B]$, we have

$$\inf_{\Delta_2} S(f, \Delta_2) \geq \inf_{\Delta} S(f, \Delta) - \inf_{\Delta_1} S(f, \Delta_1),$$

and so

$$\inf_{\Delta_1} S(f, \Delta_1) + \inf_{\Delta_2} S(f, \Delta_2) \geq \inf_{\Delta} S(f, \Delta). \tag{21}$$

The assertion (19) now follows on combining (20) and (21). ◯

Next, we investigate the question of combining two intervals of integration.

THEOREM 6G. *Suppose that $A, B, C \in \mathbb{R}$ and $A < C < B$. Suppose further that $f \in \mathcal{R}([A, C])$ and $f \in \mathcal{R}([C, B])$. Then $f \in \mathcal{R}([A, B])$. Furthermore,*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^C f(x)\,\mathrm{d}x + \int_C^B f(x)\,\mathrm{d}x.$$

Proof. Since $f \in \mathcal{R}([A, C])$ and $f \in \mathcal{R}([C, B])$, it follows from Theorem 6D that given any $\epsilon > 0$, there exist dissections Δ_1 and Δ_2 of $[A, C]$ and $[C, B]$ respectively such that

$$S(f, \Delta_1) - s(f, \Delta_1) < \frac{\epsilon}{2} \qquad \text{and} \qquad S(f, \Delta_2) - s(f, \Delta_2) < \frac{\epsilon}{2}. \tag{22}$$

Clearly $\Delta = \Delta_1 \cup \Delta_2$ is a dissection of $[A, B]$. Furthermore,

$$S(f, \Delta) = S(f, \Delta_1) + S(f, \Delta_2) \qquad \text{and} \qquad s(f, \Delta) = s(f, \Delta_1) + s(f, \Delta_2).$$

Hence

$$S(f, \Delta) - s(f, \Delta) = (S(f, \Delta_1) - s(f, \Delta_1)) + (S(f, \Delta_2) - s(f, \Delta_2)) < \epsilon,$$

in view of (22). It now follows from Theorem 6D that $f \in \mathcal{R}([A, B])$. The last assertion now follows immediately from Theorem 6F. ◯

Finally, we consider the question of altering the value of the function at a finite number of points. The following result may be applied a finite number of times.

THEOREM 6H. *Suppose that $f \in \mathcal{R}([A,B])$, where $A,B \in \mathbb{R}$ and $A < B$. Suppose further that the real number $C \in [A,B]$, and that $f(x) = g(x)$ for every $x \in [A,B]$ except possibly at $x = C$. Then $g \in \mathcal{R}([A,B])$, and*

$$\int_A^B f(x)\,\mathrm{d}x = \int_A^B g(x)\,\mathrm{d}x.$$

Proof. Write $h(x) = f(x) - g(x)$ for every $x \in [A,B]$. We shall show that

$$\int_A^B h(x)\,\mathrm{d}x = 0.$$

Note that $h(x) = 0$ whenever $x \neq C$. The case $h(C) = 0$ is trivial, so we assume, without loss of generality, that $h(C) \neq 0$. Given any $\epsilon > 0$, we shall choose a dissection Δ of $[A,B]$ such that C is not one of the dissection points and such that the subinterval containing C has length less than $\epsilon/|h(C)|$. Since $-|h(C)| \leq h(C) \leq |h(C)|$, it is easy to check that

$$S(h,\Delta) \leq |h(C)|\frac{\epsilon}{|h(C)|} < \epsilon \qquad \text{and} \qquad s(h,\Delta) \geq -|h(C)|\frac{\epsilon}{|h(C)|} > -\epsilon.$$

Hence

$$-\epsilon < I^-(h,A,B) \leq I^+(h,A,B) < \epsilon.$$

Note now that $\epsilon > 0$ is arbitrary, and the terms $I^-(h,A,B)$ and $I^+(h,A,B)$ are independent of ϵ. It follows that we must have $I^-(h,A,B) = I^+(h,A,B) = 0$. This completes the proof. ◯

6.3. Sufficient Conditions for Integrability

There are a few conditions that guarantee Riemann integrability. Here we shall study two such instances.

Definition. Suppose that $f(x)$ is a function defined on an interval I.
(1) We say that $f(x)$ is increasing in I if $f(x_1) \leq f(x_2)$ for every $x_1, x_2 \in I$ satisfying $x_1 < x_2$.
(2) We say that $f(x)$ is decreasing in I if $f(x_1) \geq f(x_2)$ for every $x_1, x_2 \in I$ satisfying $x_1 < x_2$.
(3) We say that $f(x)$ is monotonic in I if it is increasing in I or decreasing in I.

Remark. Note that a constant function on an interval I is both increasing in I and decreasing in I.

THEOREM 6J. *Suppose that a function $f(x)$ is monotonic in the closed interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A < B$. Then $f \in \mathcal{R}([A,B])$.*

Proof. The result is trivial if $f(A) = f(B)$, so we may assume that $f(A) \neq f(B)$. We may further assume, without loss of generality, that $f(x)$ is increasing in $[A,B]$, so that $f(A) < f(B)$. Given any $\epsilon > 0$, we shall consider a dissection

$$\Delta: \; A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

of $[A,B]$ such that

$$x_i - x_{i-1} < \frac{\epsilon}{f(B) - f(A)} \qquad \text{for every } i = 1, \ldots, n.$$

Since $f(x)$ is increasing in $[A,B]$, we have

$$S(f,\Delta) = \sum_{i=1}^n (x_i - x_{i-1}) f(x_i) \qquad \text{and} \qquad s(f,\Delta) = \sum_{i=1}^n (x_i - x_{i-1}) f(x_{i-1}),$$

so that

$$S(f,\Delta)-s(f,\Delta)=\sum_{i=1}^{n}(x_i-x_{i-1})(f(x_i)-f(x_{i-1}))<\frac{\epsilon}{f(B)-f(A)}\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))=\epsilon.$$

The result now follows from Theorem 6D. ○

THEOREM 6K. *Suppose that a function $f(x)$ is continuous in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Then $f\in\mathcal{R}([A,B])$.*

Here we need the idea of uniformity in continuity.

DEFINITION. A function $f(x)$ is said to be uniformly continuous in an interval I if, given any $\epsilon>0$, there exists $\delta>0$ such that

$$|f(x)-f(y)|<\epsilon \qquad \text{whenever } x,y\in I \text{ and } |x-y|<\delta.$$

It is easy to show that if $f(x)$ is uniformly continuous in an interval I, then it is continuous in I. The converse is not true, as can be seen from the following example.

EXAMPLE 6.3.1. Consider the function $f(x)=1/x$ in the open interval $(0,1)$. Then given any $\delta>0$, there exists $n\in\mathbb{N}$ such that $n^2>\delta^{-1}$. Note now that

$$\left|f\left(\frac{1}{n}\right)-f\left(\frac{1}{n+1}\right)\right|=1 \qquad \text{and} \qquad \left|\frac{1}{n}-\frac{1}{n+1}\right|=\frac{1}{n(n+1)}<\frac{1}{n^2}<\delta.$$

THEOREM 6L. *Suppose that a function $f(x)$ is continuous in the closed interval $[A,B]$, where $A,B\in\mathbb{R}$ and $A<B$. Then $f(x)$ is uniformly continuous in $[A,B]$.*

PROOF. Suppose on the contrary that $f(x)$ is not uniformly continuous in $[A,B]$. Then there exists $\epsilon>0$ such that for every $n\in\mathbb{N}$, there exist $x_n,y_n\in[A,B]$ such that

$$|x_n-y_n|<\frac{1}{n} \qquad \text{and} \qquad |f(x_n)-f(y_n)|\geq\epsilon.$$

The sequence x_n is clearly bounded, and so has a convergent subsequence x_{n_p}. Suppose that $x_{n_p}\to c$ as $p\to\infty$. Then

$$|y_{n_p}-c|\leq|x_{n_p}-y_{n_p}|+|x_{n_p}-c|\to 0 \qquad \text{as } p\to\infty,$$

so that $y_{n_p}\to c$ as $p\to\infty$. Suppose first of all that $c\in(A,B)$. Since $f(x)$ is continuous in $[A,B]$, it is continuous at c, and so $f(x_{n_p})\to f(c)$ and $f(y_{n_p})\to f(c)$ as $p\to\infty$. Note now that

$$|f(x_{n_p})-f(y_{n_p})|\leq|f(x_{n_p})-f(c)|+|f(y_{n_p})-f(c)|.$$

This implies that $|f(x_{n_p})-f(y_{n_p})|\to 0$ as $p\to\infty$, clearly a contradiction. If $c=A$ or $c=B$, then there is only one-sided continuity at c, and the proof requires minor modification. ○

PROOF OF THEOREM 6K. In view of Theorem 6L, given any $\epsilon>0$, there exists $\delta>0$ such that

$$|f(x)-f(y)|<\frac{\epsilon}{B-A} \qquad \text{whenever } x,y\in[A,B] \text{ and } |x-y|<\delta.$$

We now consider a dissection

$$\Delta:\ A=x_0<x_1<x_2<\ldots<x_n=B$$

of $[A, B]$ such that

$$x_i - x_{i-1} < \delta \qquad \text{for every } i = 1, \ldots, n.$$

Then

$$\begin{aligned} S(f, \Delta) - s(f, \Delta) &= \sum_{i=1}^{n} (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1}, x_i]} f(x) - \inf_{x \in [x_{i-1}, x_i]} f(x) \right) \\ &\le \frac{\epsilon}{B - A} \sum_{i=1}^{n} (x_i - x_{i-1}) = \epsilon. \end{aligned}$$

The result now follows from Theorem 6D. ◯

6.4. Integration as the Inverse of Differentiation

In this section, we shall establish the principle that if we can find an indefinite integral, then we can calculate definite integrals. However, we shall first establish some properties of the indefinite integral.

THEOREM 6M. *Suppose that $f \in \mathcal{R}([A, B])$, where $A, B \in \mathbb{R}$ and $A < B$. Suppose further that*

$$F(x) = \int_A^x f(t) \, \mathrm{d}t$$

for every $x \in [A, B]$. Then the following assertions hold:
(a) The function $F(x)$ is continuous in $[A, B]$.
(b) For every $a \in (A, B)$ such that $f(x)$ is continuous at $x = a$, we have $F'(a) = f(a)$.

Proof. (a) Suppose that $a \in (A, B)$. Then

$$F(a + h) - f(a) = \int_a^{a+h} f(t) \, \mathrm{d}t.$$

If $h > 0$, then it follows from Theorem 6E(d) that

$$h \inf_{t \in [A, B]} f(t) \le \int_a^{a+h} f(t) \, \mathrm{d}t \le h \sup_{t \in [A, B]} f(t),$$

so that $F(a + h) - F(a) \to 0$ as $h \to 0+$. An essentially similar argument holds for $h < 0$ and $h \to 0-$. The argument has to be slightly modified if $a = A$ or $a = B$.

(b) Suppose first of all that $h > 0$. Then it follows from Theorem 6E(d) that

$$h \inf_{t \in [a, a+h]} f(t) \le \int_a^{a+h} f(t) \, \mathrm{d}t \le h \sup_{t \in [a, a+h]} f(t),$$

so that

$$\inf_{t \in [a, a+h]} f(t) \le \frac{F(a + h) - F(a)}{h} \le \sup_{t \in [a, a+h]} f(t).$$

If $f(x)$ is continuous at $x = a$, then

$$\inf_{t \in [a, a+h]} f(t) \to f(a) \qquad \text{and} \qquad \sup_{t \in [a, a+h]} f(t) \to f(a) \qquad \text{as } h \to 0+,$$

so that

$$\frac{F(a+h)-F(a)}{h} \to f(a) \qquad \text{as } h \to 0+.$$

An essentially similar argument holds for $h<0$ and $h \to 0-$. $\bigcirc$

THEOREM 6N. *Suppose that $f(x)$ is continuous in the interval $[A,B]$, where $A,B \in \mathbb{R}$ and $A<B$. Suppose further that $\phi'(x) = f(x)$ for every $x \in [A,B]$. Then for every $x \in [A,B]$, we have*

$$\int_A^x f(t)\,\mathrm{d}t = \phi(x) - \phi(A).$$

PROOF. It follows from Theorem 6M that $F'(x) - \phi'(x) = 0$ for every $x \in (A,B)$, so that $F(x) - \phi(x)$ is constant in $[A,B]$ by Theorem 5H(a). Since $F(A) = 0$, we must have $F(x) = \phi(x) - \phi(a)$ for every $x \in [A,B]$. $\bigcirc$

6.5. An Important Example

In this section, we shall find a function that is not Riemann integrable. Consider the function

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

We know from Theorem 1D that in any open interval, there are rational numbers and irrational numbers. It follows that in any interval $[\alpha,\beta]$, where $\alpha<\beta$, we have

$$\inf_{x\in[\alpha,\beta]} g(x) = 0 \qquad \text{and} \qquad \sup_{x\in[\alpha,\beta]} g(x) = 1.$$

It follows that for every dissection Δ of $[0,1]$, we have

$$s(g,\Delta) = 0 \qquad \text{and} \qquad S(g,\Delta) = 1,$$

so that

$$I^-(g,0,1) = 0 \neq 1 = I^+(g,0,1).$$

It follows that $g(x)$ is not Riemann integrable over the closed interval $[0,1]$.

Note, on the other hand, that the rational numbers in $[0,1]$ are countable, while the irrational numbers in $[0,1]$ are not countable. In the sense of cardinality, there are far more irrational numbers than rational numbers in $[0,1]$. However, the definition of the Riemann integral does not highlight this inequality.

We wish therefore to develop a theory of integration more general than Riemann integration. This is the motivation for the Lebesgue integral.

Problems for Chapter 6

1. Calculate the integral $\int_0^1 x\,\mathrm{d}x$ by dissecting the interval $[0,1]$ into equal parts.

2. Calculate the integral $\int_A^B x^k\,\mathrm{d}x$, where $k>0$ is fixed, by dissecting the interval $[A,B]$ into n parts in geometric progression, so that $A<Aq<Aq^2<\ldots<Aq^n=B$.

3. a) By using the method of Problem 2, prove that $\displaystyle\int_1^2 \frac{1}{x^2}\,\mathrm{d}x=\frac{1}{2}$.
   b) Deduce that $\displaystyle\lim_{n\to\infty} n\left(\frac{1}{(n+1)^2}+\frac{1}{(n+2)^2}+\ldots+\frac{1}{(2n)^2}\right)=\frac{1}{2}$.

4. Calculate the integral $\int_0^\alpha \sin x\,\mathrm{d}x$ by dissecting the interval $[0,\alpha]$ into equal parts.

5. Consider the function $f(x)=1/x$ in the closed interval $[1,2]$. For every $n\in\mathbb{N}$, let Δ_n denote the dissection of the interval $[1,2]$ into n subintervals of equal length.
   a) Find $s(f,\Delta_n)$ and $S(f,\Delta_n)$, and show that
   $$S(f,\Delta_n)-s(f,\Delta_n)=\frac{1}{2n}.$$
   b) Show that $f\in\mathcal{R}([1,2])$.
   c) Explain why the value of the integral is equal to
   $$\lim_{n\to\infty}\left(\frac{1}{n+1}+\frac{1}{n+2}+\ldots+\frac{1}{2n}\right).$$

6. In this question, we shall try to verify from the definition of the Riemann integral that
   $$\int_0^1 f(x)\,\mathrm{d}x=\frac{2}{\pi},\qquad\text{where } f(x)=\cos\frac{\pi x}{2}.$$
   For every $n\in\mathbb{N}$, let Δ_n denote the dissection of the interval $[0,1]$ into n subintervals of equal length.
   a) Find $s(f,\Delta_n)$ and $S(f,\Delta_n)$, and show that
   $$S(f,\Delta_n)-s(f,\Delta_n)=\frac{1}{n}.$$
   b) Show that $f\in\mathcal{R}([0,1])$.
   c) Explain why
   $$\int_0^1 f(x)\,\mathrm{d}x=\lim_{n\to\infty}S(f,\Delta_n).$$
   d) Note that $\cos(k-1)\theta=\Re(\mathrm{e}^{\mathrm{i}(k-1)\theta})$, so that $S(f,\Delta_n)$ is the real part of a geometric series. Sum the geometric series and show that
   $$S(f,\Delta_n)=\frac{1}{n}\Re\left(\frac{1-\mathrm{e}^{\mathrm{i}n\theta}}{1-\mathrm{e}^{\mathrm{i}\theta}}\right)=\frac{1}{n}\Re\left(\frac{1-\mathrm{i}}{1-\mathrm{e}^{\mathrm{i}\theta}}\right)=\frac{\theta}{\pi}+\frac{\theta\sin\theta}{\pi(1-\cos\theta)},\qquad\text{where } \theta=\frac{\pi}{2n}.$$
   e) Explain why
   $$\lim_{n\to\infty}S(f,\Delta_n)=\frac{2}{\pi}.$$

7. Suppose that a function $f(x)$ is bounded on the closed interval $[A, B]$, where $A, B \in \mathbb{R}$ and $A < B$.
   a) Show that for any closed interval $I \subseteq [A, B]$,

$$\sup_{x \in I} |f(x)| - \inf_{x \in I} |f(x)| \le \sup_{x \in I} f(x) - \inf_{x \in I} f(x).$$

   b) Show that for every dissection Δ of the interval $[A, B]$,

$$S(|f|, \Delta) - s(|f|, \Delta) \le S(f, \Delta) - s(f, \Delta).$$

   c) Show that if $f \in \mathcal{R}([A, B])$, then $|f| \in \mathcal{R}([A, B])$.
   d) Note that $-|f(x)| \le f(x) \le |f(x)|$ for every $x \in [A, B]$. Use this to show that if $f \in \mathcal{R}([A, B])$, then

$$\left| \int_A^B f(x) \, \mathrm{d}x \right| \le \int_A^B |f(x)| \, \mathrm{d}x.$$

8. Suppose that $f, g \in \mathcal{R}([A, B])$, where $A, B \in \mathbb{R}$ and $A < B$.
   a) Show that $f^2 \in \mathcal{R}([A, B])$.
   b) Use part (a) to deduce that $fg \in \mathcal{R}([A, B])$.
   c) Suppose further that $m \le f(x) \le M$ and $g(x) \ge 0$ for every $x \in [A, B]$. Show that

$$m \int_A^B g(x) \, \mathrm{d}x \le \int_A^B f(x)g(x) \, \mathrm{d}x \le M \int_A^B g(x) \, \mathrm{d}x.$$

   d) By considering the integral

$$\int_A^B (\lambda f(x) + \mu g(x))^2 \, \mathrm{d}x$$

   for suitable constants λ and μ, establish Schwarz's inequality

$$\left(\int_A^B f(x)g(x) \, \mathrm{d}x \right)^2 \le \left(\int_A^B f^2(x) \, \mathrm{d}x \right) \left(\int_A^B g^2(x) \, \mathrm{d}x \right).$$