FUNDAMENTALS OF ANALYSIS

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Chapter 7

FURTHER TREATMENT OF LIMITS

7.1. Upper and Lower Limits of a Real Sequence

Suppose that x_n is a sequence of real numbers bounded above. For every $n \in \mathbb{N}$, let

$$K_n = \sup\{x_n, x_{n+1}, x_{n+2}, \ldots\}.$$

Then K_n is a decreasing sequence, and converges as $n \to \infty$ if it is bounded below.

Definition. Suppose that x_n is a sequence of real numbers bounded above. The number

$$\Lambda = \lim_{n\to\infty} \left(\sup_{r\ge n} x_r \right),$$

if it exists, is called the upper limit of x_n, and denoted by

$$\Lambda = \limsup_{n\to\infty} x_n \qquad \text{or} \qquad \Lambda = \overline{\lim_{n\to\infty}}\, x_n.$$

Definition. Suppose that x_n is a sequence of real numbers bounded below. The number

$$\lambda = \lim_{n\to\infty} \left(\inf_{r\ge n} x_r \right),$$

if it exists, is called the lower limit of x_n, and denoted by

$$\lambda = \liminf_{n\to\infty} x_n \qquad \text{or} \qquad \lambda = \underline{\lim_{n\to\infty}}\, x_n.$$

REMARK. It is obvious that $\lambda \leq \Lambda$, since the infimum of a bounded set of real number never exceeds the corresponding supremum.

EXAMPLE 7.1.1. For the sequence $x_n = (-1)^n$, we have $\Lambda = 1$ and $\lambda = -1$.

EXAMPLE 7.1.2. For the sequence $x_n = n/(n+1)$, we have $\Lambda = \lambda = 1$.

EXAMPLE 7.1.3. For the sequence $x_n = n(1+(-1)^n)$, we have $\lambda = 0$ and Λ does not exist.

EXAMPLE 7.1.4. For the sequence $x_n = \sin \frac{1}{2}n\pi$, we have $\Lambda = 1$ and $\lambda = -1$.

THEOREM 7A. *Suppose that x_n is a sequence of real numbers. Then the following two statements are equivalent:*
(a) We have $\Lambda = \limsup_{n\to\infty} x_n$.
(b) For every $\epsilon > 0$, we have
(i) $x_n < \Lambda + \epsilon$ for all sufficiently large $n \in \mathbb{N}$; and
(ii) $x_n > \Lambda - \epsilon$ for infinitely many $n \in \mathbb{N}$.

PROOF. ((a)⇒(b)) Suppose that

$$\Lambda = \limsup_{n\to\infty} x_n = \lim_{n\to\infty} K_n, \qquad \text{where} \qquad K_n = \sup_{r\geq n} x_r.$$

Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $|K_N - \Lambda| < \epsilon$, so that in particular, $K_N < \Lambda + \epsilon$. It follows that $x_n < \Lambda + \epsilon$ for every $n \geq N$, giving (i). On the other hand, for every $\epsilon > 0$ and every $N \in \mathbb{N}$, there exists $n \geq N$ such that $x_n > K_N - \epsilon$. Clearly $K_N \geq \Lambda$ for every $N \in \mathbb{N}$, giving (ii).

((b)⇒(a)) Given any $\epsilon > 0$, it follows from (i) that $K_n \leq \Lambda + \epsilon$ for all sufficiently large $n \in \mathbb{N}$, and from (ii) that $K_n > \Lambda - \epsilon$ for every $n \in \mathbb{N}$. Clearly $K_n \to \Lambda$ as $n \to \infty$. ◯

Similarly, we have the following result.

THEOREM 7B. *Suppose that x_n is a sequence of real numbers. Then the following two statements are equivalent:*
(a) We have $\lambda = \liminf_{n\to\infty} x_n$.
(b) For every $\epsilon > 0$, we have
(i) $x_n > \lambda - \epsilon$ for all sufficiently large $n \in \mathbb{N}$; and
(ii) $x_n < \lambda + \epsilon$ for infinitely many $n \in \mathbb{N}$.

We now establish the following important result.

THEOREM 7C. *Suppose that x_n is a sequence of real numbers. Then*

$$\lim_{n\to\infty} x_n = \ell \qquad \textit{if and only if} \qquad \limsup_{n\to\infty} x_n = \liminf_{n\to\infty} x_n = \ell.$$

PROOF. (⇒) Suppose that $x_n \to \ell$ as $n \to \infty$. Then the upper and lower limits of the sequence x_n clearly exist, since x_n is bounded in this case. Also, given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\ell - \epsilon < x_n < \ell + \epsilon$ for every $n \geq N$. The conclusion follows immediately from Theorems 7A and 7B.

(⇐) Suppose that the upper and lower limits are both equal to ℓ. Then it follows from Theorem 7A that $x_n < \ell + \epsilon$ for all sufficiently large $n \in \mathbb{N}$, and from Theorem 7B that $x_n > \ell - \epsilon$ for all sufficiently large $n \in \mathbb{N}$. Hence $|x_n - \ell| < \epsilon$ for all sufficiently large $n \in \mathbb{N}$, whence $x_n \to \ell$ as $n \to \infty$. ◯

7.2. Double and Repeated Limits

We shall consider a double sequence z_{mn} of complex numbers, represented by a doubly infinite array

$$\begin{matrix} z_{11} & z_{12} & z_{13} & \cdots \\ z_{21} & z_{22} & z_{23} & \cdots \\ z_{31} & z_{32} & z_{33} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{matrix}$$

of complex numbers. More precisely, a double sequence of complex numbers is simply a mapping from $\mathbb{N} \times \mathbb{N}$ to $\mathbb{C}$.

Definition. We say that a double sequence z_{mn} converges to a finite limit $z \in \mathbb{C}$, denoted by $z_{mn} \to z$ as $m, n \to \infty$ or by

$$\lim_{m,n\to\infty} z_{mn} = z,$$

if, given any $\epsilon > 0$, there exists $N = N(\epsilon) \in \mathbb{R}$, depending on ϵ, such that $|z_{mn} - z| < \epsilon$ whenever $m, n > N$. Furthermore, we say that a double sequence z_{mn} is convergent if it converges to some finite limit z as $m, n \to \infty$, and that a double sequence z_{mn} is divergent if it is not convergent.

Example 7.2.1. For the double sequence

$$z_{mn} = \frac{1}{m+n},$$

we have $z_{mn} \to 0$ as $m, n \to \infty$.

Example 7.2.2. The double sequence

$$z_{mn} = \frac{m}{m+n}$$

does not converge to a finite limit as $m, n \to \infty$. Note that for all sufficiently large $m, n \in \mathbb{N}$ with $m = n$, we have $z_{mn} = \frac{1}{2}$, whereas for all sufficiently large $m, n \in \mathbb{N}$ with $m = 2n$, we have $z_{mn} = \frac{2}{3}$.

The question we want to study is the relationship, if any, between the following three limiting processes when applied to a double sequence z_{mn} of complex numbers:
- $m, n \to \infty$.
- $n \to \infty$ followed by $m \to \infty$.
- $m \to \infty$ followed by $n \to \infty$.

THEOREM 7D. *Suppose that a double sequence z_{mn} satisfies the following conditions:*
(a) The double limit $\displaystyle\lim_{m,n\to\infty} z_{mn}$ exists.
(b) For every $m \in \mathbb{N}$, the limit $\displaystyle\lim_{n\to\infty} z_{mn}$ exists.

Then the repeated limit $\displaystyle\lim_{m\to\infty}\left(\lim_{n\to\infty} z_{mn}\right)$ exists, and is equal to the double limit $\displaystyle\lim_{m,n\to\infty} z_{mn}$.

Remark. We need to make the assumption (b), as it does not necessarily follow from assumption (a). Consider, for example, the double sequence

$$z_{mn} = \frac{(-1)^n}{m}.$$

Proof of Theorem 7D. Suppose that $z_{mn} \to z$ as $m, n \to \infty$. Suppose also that for every $m \in \mathbb{N}$, $z_{mn} \to \zeta_m$ as $n \to \infty$. We need to show that $\zeta_m \to z$ as $m \to \infty$. Given any $\epsilon > 0$, there exists $N \in \mathbb{R}$ such that

$$|z_{mn} - z| < \frac{\epsilon}{2} \qquad \text{whenever } m, n > N.$$

On the other hand, given any $m \in \mathbb{N}$, there exists $M(m) \in \mathbb{R}$ such that

$$|z_{mn} - \zeta_m| < \frac{\epsilon}{2} \qquad \text{whenever } n > M(m).$$

Now let $m > N$. Then choosing $n > \max\{N, M(m)\}$, we have

$$|\zeta_m - z| \le |z_{mn} - \zeta_m| + |z_{mn} - z| < \epsilon.$$

Hence $\zeta_m \to z$ as $m \to \infty$. ◯

We immediately have the following generalization.

THEOREM 7E. *Suppose that a double sequence z_{mn} satisfies the following conditions:*
(a) The double limit $\displaystyle\lim_{m,n\to\infty} z_{mn}$ exists.
(b) For every $m \in \mathbb{N}$, the limit $\displaystyle\lim_{n\to\infty} z_{mn}$ exists.
(c) For every $n \in \mathbb{N}$, the limit $\displaystyle\lim_{m\to\infty} z_{mn}$ exists.

Then the repeated limits $\displaystyle\lim_{m\to\infty}\left(\lim_{n\to\infty} z_{mn}\right)$ and $\displaystyle\lim_{n\to\infty}\left(\lim_{m\to\infty} z_{mn}\right)$ exist, and are both equal to the double limit $\displaystyle\lim_{m,n\to\infty} z_{mn}$.

We can further generalize the above to a result concerning series.

Definition. Suppose that z_{mn} is a double sequence of complex numbers. For every $m, n \in \mathbb{N}$, let

$$s_{mn} = \sum_{i=1}^{m} \sum_{j=1}^{n} z_{ij}.$$

If the double sequence $s_{mn} \to s$ as $m, n \to \infty$, then we say that the double series

$$\sum_{m,n=1}^{\infty} z_{mn}$$

is convergent, with sum s.

THEOREM 7F. *Suppose that a double sequence z_{mn} satisfies the following conditions:*
(a) The double series $\displaystyle\sum_{m,n=1}^{\infty} z_{mn}$ is convergent, with sum s.
(b) For every $m \in \mathbb{N}$, the series $\displaystyle\sum_{n=1}^{\infty} z_{mn}$ is convergent.
(c) For every $n \in \mathbb{N}$, the series $\displaystyle\sum_{m=1}^{\infty} z_{mn}$ is convergent.

Then the repeated series $\displaystyle\sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty} z_{mn}\right)$ and $\displaystyle\sum_{n=1}^{\infty}\left(\sum_{m=1}^{\infty} z_{mn}\right)$ are both convergent, with sum s.

7.3. Infinite Products

An infinite product is an expression of the form

$$(1+z_1)(1+z_2)(1+z_3)\ldots$$

with an infinitude of factors. We denote this by

$$\prod_{n=1}^{\infty}(1+z_n). \tag{1}$$

We also make the natural assumption that $z_n \neq -1$ for any $n \in \mathbb{N}$.

For every $N \in \mathbb{N}$, let

$$p_N = \prod_{n=1}^{N}(1+z_n) = (1+z_1)\ldots(1+z_N).$$

We shall call p_N the N-th partial product of the infinite product (1).

DEFINITION. If the sequence p_N converges to a non-zero limit p as $N \to \infty$, then we say that the infinite product (1) converges to p and write

$$\prod_{n=1}^{\infty}(1+z_n) = p.$$

In this case, we sometimes simply say that the infinite product (1) is convergent. On the other hand, if the sequence p_N does not cionverge to a non-zero limit as $N \to \infty$, then we say that the infinite product (1) is divergent. In particular, if $p_N \to 0$ as $N \to \infty$, then we say that the infinite product (1) diverges to zero.

Let us first examine the special case when all the terms z_n are real.

THEOREM 7G. *Suppose that $a_n \geq 0$ for every $n \in \mathbb{N}$. Then the infinite product*

$$\prod_{n=1}^{\infty}(1+a_n)$$

is convergent if and only if the series

$$\sum_{n=1}^{\infty} a_n$$

is convergent.

PROOF. Let s_N be the N-th partial sum of the series. Since $a_n \geq 0$ for every $n \in \mathbb{N}$, the sequences s_N and p_N are both increasing. On the other hand, note that $1 + a \leq \mathrm{e}^a$ for every $a \geq 0$. It follows that for every $N \in \mathbb{N}$, we have

$$a_1 + \ldots + a_N \leq (1+a_1)\ldots(1+a_N) \leq \mathrm{e}^{a_1+\ldots+a_N},$$

so that $s_N \leq p_N \leq \mathrm{e}^{s_N}$. It follows that the sequences s_N and p_N are bounded or unbounded together. The result follows from Theorem 2E. ◯

If $a_n \le 0$ for every $n \in \mathbb{N}$, then we write $a_n = -b_n$ and consider the infinite product

$$\prod_{n=1}^{\infty}(1-b_n). \tag{2}$$

THEOREM 7H. *Suppose that $0 \le b_n < 1$ for every $n \in \mathbb{N}$. Then the infinite product (2) is convergent if and only if the series*

$$\sum_{n=1}^{\infty} b_n \tag{3}$$

is convergent.

This follows immediately from the following two results.

THEOREM 7J. *Suppose that $0 \le b_n < 1$ for every $n \in \mathbb{N}$. Suppose further that the series (3) is convergent. Then the infinite product (2) is convergent.*

THEOREM 7K. *Suppose that $0 \le b_n < 1$ for every $n \in \mathbb{N}$. Suppose further that the series (3) is divergent. Then the infinite product (2) diverges to zero.*

Proof of Theorem 7J. Since the series (3) is convergent, there exists $M \in \mathbb{N}$ such that

$$\sum_{n=M+1}^{\infty} b_n < \frac{1}{2}.$$

Hence for every $N > M$, we have

$$(1-b_{M+1})(1-b_{M+2})\ldots(1-b_N) \ge 1 - b_{M+1} - b_{M+2} - \ldots - b_N > \frac{1}{2}.$$

It follows that the sequence p_N is a decreasing sequence bounded below by $\frac{1}{2}p_M \ne 0$, so that p_N converges to a non-zero limit as $N \to \infty$. ◯

Proof of Theorem 7K. Note that $1 - b \le \mathrm{e}^{-b}$ whenever $0 \le b < 1$. It follows that for every $N \in \mathbb{N}$, we have

$$0 \le (1-b_1)\ldots(1-b_N) \le \mathrm{e}^{-b_1-\ldots-b_N}.$$

Note now that $\mathrm{e}^{-b_1-\ldots-b_N} \to 0$ as $N \to \infty$. The result follows from the Squeezing principle. ◯

We now investigate the general case, where $z_n \in \mathbb{C} \setminus \{-1\}$ for every $n \in \mathbb{N}$.

Definition. The infinite product (1) is said to be absolutely convergent if the infinite product

$$\prod_{n=1}^{\infty}(1+|z_n|)$$

is convergent.

The following result is an obvious consequence of Theorem 7G.

THEOREM 7L. *The infinite product* (1) *is absolutely convergent if and only if the series*

$$\sum_{n=1}^{\infty} z_n \tag{4}$$

is absolutely convergent.

On the other hand, as in series, we have the following result.

THEOREM 7M. *Suppose that the infinite product* (1) *is absolutely convergent. Then it is also convergent.*

PROOF. For every $N \in \mathbb{N}$, let

$$p_N = \prod_{n=1}^{N} (1 + z_n) \qquad \text{and} \qquad P_N = \prod_{n=1}^{N} (1 + |z_n|).$$

If $N \geq 2$, then

$$p_N - p_{N-1} = (1 + z_1) \ldots (1 + z_{N-1}) z_N \qquad \text{and} \qquad P_N - P_{N-1} = (1 + |z_1|) \ldots (1 + |z_{N-1}|) |z_N|,$$

so that

$$|p_N - p_{N-1}| \leq P_N - P_{N-1}. \tag{5}$$

If we write $p_0 = P_0 = 0$, then (5) holds also for $N = 1$. Furthermore, for every $N \in \mathbb{N}$, we have

$$p_N = \sum_{n=1}^{N} (p_n - p_{n-1}) \qquad \text{and} \qquad P_N = \sum_{n=1}^{N} (P_n - P_{n-1}).$$

Since P_N converges as $N \to \infty$, it follows from the Comparison test that p_N converges as $N \to \infty$. It remains to show that p_N does not converge to 0 as $N \to \infty$. Note from Theorem 7L that the series (4) is absolutely convergent, so that $z_n \to 0$ as $n \to \infty$, and so $1 + z_n \to 1$ as $n \to \infty$. Hence the series

$$\sum_{n=1}^{\infty} \left| \frac{z_n}{1 + z_n} \right|$$

is convergent, and so it follows from Theorem 7L that the infinite product

$$\prod_{n=1}^{\infty} \left(1 + \left| -\frac{z_n}{1 + z_n} \right| \right) \tag{6}$$

is convergent. Repeating the first part of our argument on the infinite product (6), we conclude that the sequence

$$\prod_{n=1}^{N} \left(1 - \frac{z_n}{1 + z_n} \right)$$

is convergent as $N \to \infty$. Note now that this product is precisely $1/p_N$. ◯

7.4. Double Integrals

The purpose of this last section is to give a sketch of the proof of the following result concerning double integrals.

THEOREM 7N. *Suppose that a function $f(x,y)$ is continuous in a closed rectangle $[A,B]\times[C,D]$, where $A,B,C,D\in\mathbb{R}$ satisfy $A<B$ and $C<D$. Then the double integrals*

$$\int_A^B \mathrm{d}x \int_C^D f(x,y)\,\mathrm{d}y \qquad \textit{and} \qquad \int_C^D \mathrm{d}y \int_A^B f(x,y)\,\mathrm{d}x$$

exist in the sense of Riemann, and are equal to each other.

SKETCH OF PROOF. The idea is to first show that $f(x,y)$ is uniformly continuous in the rectangle $[A,B]\times[C,D]$, in the spirit of Theorem 6L. Using the uniform continuity, one can then show that the function

$$\phi(y)=\int_A^B f(x,y)\,\mathrm{d}x$$

is continuous in the closed interval $[C,D]$. It follows from Theorem 6K that the integral

$$\int_C^D \mathrm{d}y \int_A^B f(x,y)\,\mathrm{d}x$$

exists. Similarly the other integral

$$\int_A^B \mathrm{d}x \int_C^D f(x,y)\,\mathrm{d}y$$

exists. To show that the two integrals are equal, we make use of the uniform continuity again. Given any $\epsilon>0$, there exist dissections $A=x_0<x_1<\ldots<x_k=B$ and $C=y_0<y_1<\ldots<y_n=D$ of the intervals $[A,B]$ and $[C,D]$ respectively such that

$$M_{ij}-m_{ij}<\frac{\epsilon}{(B-A)(D-C)} \qquad \text{for every } i=1,\ldots,k \text{ and } j=1,\ldots,n,$$

where

$$M_{ij}=\sup_{\substack{x_{i-1}\le x\le x_i\\ y_{j-1}\le y\le y_j}} f(x,y) \qquad \text{and} \qquad m_{ij}=\inf_{\substack{x_{i-1}\le x\le x_i\\ y_{j-1}\le y\le y_j}} f(x,y).$$

For every $i=1,\ldots,k$ and $j=1,\ldots,n$, we have

$$m_{ij}(x_i-x_{i-1})\le\int_{x_{i-1}}^{x_i} f(x,y)\,\mathrm{d}x\le M_{ij}(x_i-x_{i-1}) \qquad \text{for every } y\in[y_{j-1},y_j],$$

so that

$$m_{ij}(x_i-x_{i-1})(y_j-y_{j-1})\le\int_{y_{j-1}}^{y_j}\mathrm{d}y\int_{x_{i-1}}^{x_i} f(x,y)\,\mathrm{d}x\le M_{ij}(x_i-x_{i-1})(y_j-y_{j-1}).$$

Summing over all i and j, we obtain

$$\sum_{i=1}^k\sum_{j=1}^n m_{ij}(x_i-x_{i-1})(y_j-y_{j-1})\le\int_C^D \mathrm{d}y\int_A^B f(x,y)\,\mathrm{d}x\le\sum_{i=1}^k\sum_{j=1}^n M_{ij}(x_i-x_{i-1})(y_j-y_{j-1}).$$

A similar argument gives

$$\sum_{i=1}^{k}\sum_{j=1}^{n} m_{ij}(x_i - x_{i-1})(y_j - y_{j-1}) \le \int_A^B \mathrm{d}x \int_C^D f(x,y)\,\mathrm{d}y \le \sum_{i=1}^{k}\sum_{j=1}^{n} M_{ij}(x_i - x_{i-1})(y_j - y_{j-1}).$$

Hence

$$\left| \int_C^D \mathrm{d}y \int_A^B f(x,y)\,\mathrm{d}x - \int_A^B \mathrm{d}x \int_C^D f(x,y)\,\mathrm{d}y \right| \le \sum_{i=1}^{k}\sum_{j=1}^{n} (M_{ij} - m_{ij})(x_i - x_{i-1})(y_j - y_{j-1}) < \epsilon.$$

The result now follows since $\epsilon > 0$ is arbitrary and the left hand side is independent of ϵ. ◯

It turns out that the conclusion of Theorem 7N may still hold even if the function $f(x,y)$ is not continuous everywhere in the rectangle $[A,B] \times [C,D]$. We state without proof the following result.

THEOREM 7P. *Suppose that a function $f(x,y)$ is continuous in a closed rectangle $[A,B] \times [C,D]$, where $A,B,C,D \in \mathbb{R}$ satisfy $A < B$ and $C < D$, except possibly at points along a curve of type defined by one of the following:*
(a) $x = \alpha$ for some $\alpha \in [A,B]$.
(b) $y = \gamma$ for some $\gamma \in [C,D]$.
(c) $x = \psi(y)$ for $y \in [\gamma,\delta]$, where $C \le \gamma \le \delta \le D$ and $\psi(y)$ is strictly monotonic and continuous.
Then the conclusion of Theorem 7N holds.

Problems for Chapter 7

1. Suppose that x_n and y_n are bounded real sequences.
   a) Show that
   $$\underline{\lim}_{n\to\infty} x_n + \underline{\lim}_{n\to\infty} y_n \le \underline{\lim}_{n\to\infty} (x_n + y_n) \le \underline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n \le \overline{\lim}_{n\to\infty} (x_n + y_n) \le \overline{\lim}_{n\to\infty} x_n + \overline{\lim}_{n\to\infty} y_n.$$
   b) Find sequences x_n and y_n where equality holds nowhere in part (a).
   c) Suppose further that $x_n \ge 0$ and $y_n \ge 0$ for every $n \in \mathbb{N}$. Establish a chain of inequalities as in part (a) but with products in place of sums.
   d) Find sequences x_n and y_n where equality holds nowhere in part (c).

2. For each of the following double sequences z_{mn}, find the double limit $\lim_{m,n\to\infty} z_{mn}$ and the repeated limits $\lim_{m\to\infty}\left(\lim_{n\to\infty} z_{mn}\right)$ and $\lim_{n\to\infty}\left(\lim_{m\to\infty} z_{mn}\right)$, if they exist:
   a) $z_{mn} = \dfrac{m-n}{m+n}$
   b) $z_{mn} = \dfrac{m+n}{m^2}$
   c) $z_{mn} = \dfrac{m+n}{m^2+n^2}$
   d) $z_{mn} = (-1)^{m+n}\left(\dfrac{1}{m} + \dfrac{1}{n}\right)$
   e) $z_{mn} = \dfrac{mn}{m^2+n^2}$
   f) $z_{mn} = (-1)^{m+n}\dfrac{1}{n}\left(1 + \dfrac{1}{m}\right)$

3. Does there exist a double sequence z_{mn} such that z_{mn} converges as $m, n \to \infty$ but also that z_{mn} is not bounded? Justify your assertion.

4. Suppose that x_{mn} is a bounded double sequence of real numbers satisfying the following conditions:
   a) For every fixed $m \in \mathbb{N}$, the sequence x_{mn} is increasing in n.
   b) For every fixed $n \in \mathbb{N}$, the sequence x_{mn} is increasing in m.
   Prove that x_{mn} converges as $m, n \to \infty$.

5. Use Problem 4 to prove the Comparison test for double series: Suppose that $0 \le u_{mn} \le v_{mn}$ for every $m, n \in \mathbb{N}$. Suppose further that the double series
   $$\sum_{m,n=1}^{\infty} v_{mn}$$
   is convergent. Then the double series
   $$\sum_{m,n=1}^{\infty} u_{mn}$$
   is convergent.

6. Using ideas from the proof of the Alternating series test, prove that the infinite product
   $$\prod_{n=1}^{\infty}\left(1 + \frac{(-1)^{n-1}}{n}\right)$$
   is convergent.

7. Prove Theorem 7N.