CHAPTER 2

Goldbach's Problem

2.1. Introduction

Goldbach's problem concerns the solubility of the equation

$$n = p_1 + p_2. \tag{2.1}$$

Given any even natural number $n \in \mathbb{N}$ satisfying $n > 2$, the question is whether there are primes p_1, p_2 such that (2.1) holds.

The greatest success of the Hardy–Littlewood method is a demonstration of the ternary Goldbach problem, which concerns the solubility of the equation

$$n = p_1 + p_2 + p_3. \tag{2.2}$$

Given any odd natural number $n \in \mathbb{N}$ satisfying $n > 5$, the question is whether there are primes p_1, p_2, p_3 such that (2.2) holds. Here the difficulty is in handling the cases when the natural number $n \in \mathbb{N}$ is relatively small. In this chapter, we adapt the ideas of the original Hardy–Littlewood method and demonstrate that the ternary Goldbach problem can be solved if the odd natural number $n \in \mathbb{N}$ is sufficiently large, a result which dates back to Vinogradov in the 1930s.

THEOREM 2.1. *There exists N_0 such that for every odd natural number $n \in \mathbb{N}$ satisfying $n > N_0$, there exist primes p_1, p_2, p_3 such that $n = p_1 + p_2 + p_3$.*

Clearly the number of primes required cannot be reduced, so this result is in some sense best possible. The only weakness is that the conclusion is valid only for large odd natural numbers $n \in \mathbb{N}$.

The Hardy–Littlewood method can also be applied to study the binary Goldbach problem. We mention the result that for almost all even natural numbers $n \in \mathbb{N}$, there exist primes p_1, p_2 such that (2.1) holds. This means that for all sufficiently large N, the number $E(N)$ of exceptional even natural numbers $n \in \mathbb{N}$ such that (2.1) does not hold for any primes p_1, p_2 satisfies $E(N) = o(N)$.

Very recently, Helfgott has completely solved the ternary Goldbach problem. His technique involves lowering the value of N_0 in Theorem 2.1 as much as possible, as well as efficient computation to check the finite number of remaining cases.

For any Hardy–Littlewood approach to work, we naturally need to replace the generating function $f(\alpha)$. We write

$$f(\alpha) = \sum_{p \leqslant n} (\log p) e(\alpha p),$$

where the summation is over all positive primes not exceeding n. Here the presence of the term $(\log p)$ is not immediately obvious. Indeed, it is not absolutely necessary. However, its presence enables us to bring in the von Mangoldt function, used in the study of the distribution of primes, in a more natural way.

With this choice of the generating function $f(\alpha)$, we consider

$$\begin{aligned} R(n) &= \int_0^1 f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha = \int_0^1 \sum_{p_1\leqslant n}\sum_{p_2\leqslant n}\sum_{p_3\leqslant n}(\log p_1)(\log p_2)(\log p_3)e(\alpha(p_1+p_2+p_3-n))\,\mathrm{d}\alpha \\ &= \sum_{p_1\leqslant n}\sum_{p_2\leqslant n}\sum_{p_3\leqslant n}(\log p_1)(\log p_2)(\log p_3)\int_0^1 e(\alpha(p_1+p_2+p_3-n))\,\mathrm{d}\alpha \\ &= \sum_{\substack{p_1\leqslant n\; p_2\leqslant n\; p_3\leqslant n \\ p_1+p_2+p_3=n}}(\log p_1)(\log p_2)(\log p_3) = \sum_{\substack{p_1\;p_2\;p_3 \\ p_1+p_2+p_3=n}}(\log p_1)(\log p_2)(\log p_3). \end{aligned}$$

Note that the term $R(n)$ is a weighted count of the number of solutions of the equation $n = p_1+p_2+p_3$.

Remark. For the purposes of comparing the argument here with the argument in Chapter 1, note that we take $N = n$ in this chapter, and use n throughout to denote their common value.

As before, we write

$$R(n) = \int_{\mathfrak{M}} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha + \int_{\mathfrak{m}} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha, \tag{2.3}$$

where the two sets $\mathfrak{M}$ and $\mathfrak{m}$ are disjoint and $\mathfrak{M}\cup\mathfrak{m}$ represents a unit interval. Let B be a sufficiently large positive real number, and write

$$P = (\log n)^B. \tag{2.4}$$

For every $a, q \in \mathbb{N}$ satisfying $1 \leqslant a \leqslant q \leqslant P$ and $(a, q) = 1$, let

$$\mathfrak{M}(q, a) = \{\alpha \in \mathbb{R} : |\alpha - a/q| \leqslant Pn^{-1}\}. \tag{2.5}$$

We now write

$$\mathfrak{M} = \bigcup_{q\leqslant P}\bigcup_{\substack{a=1 \\ (a,q)=1}}^{q} \mathfrak{M}(q, a),$$

and let

$$\mathcal{U} = (Pn^{-1}, 1 + Pn^{-1}] \quad\text{and}\quad \mathfrak{m} = \mathcal{U} \setminus \mathfrak{M}.$$

To prove Theorem 2.1, it suffices to show that $R(n) > 0$ for all sufficiently large odd $n \in \mathbb{N}$.

Our strategy is to find a suitable positive value of B for which

$$\int_{\mathfrak{M}} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha \gg n^2$$

and

$$\int_{\mathfrak{m}} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha = o(n^2)$$

whenever n is odd. We do not give an explicit value for B, but will indicate the restrictions on its size throughout our discussion.

2.2. The Minor Arcs

In this section, we study the integral

$$\int_{\mathfrak{m}} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha.$$

It is easy to see that

$$\left|\int_{\mathfrak{m}} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha\right| \leqslant \int_{\mathfrak{m}} |f(\alpha)|^3\,\mathrm{d}\alpha \leqslant \left(\sup_{\alpha\in\mathfrak{m}}|f(\alpha)|\right)\int_0^1 |f(\alpha)|^2\,\mathrm{d}\alpha. \tag{2.6}$$

Our earlier argument involving Hua's lemma is now replaced by a very simple argument. We have

$$\int_0^1 |f(\alpha)|^2 \,\mathrm{d}\alpha = \int_0^1 f(\alpha)f(-\alpha)\,\mathrm{d}\alpha = \int_0^1 \sum_{p_1\leqslant n}\sum_{p_2\leqslant n}(\log p_1)(\log p_2)e(\alpha(p_1-p_2))\,\mathrm{d}\alpha \tag{2.7}$$

$$= \sum_{p_1\leqslant n}\sum_{p_2\leqslant n}(\log p_1)(\log p_2)\int_0^1 e(\alpha(p_1-p_2))\,\mathrm{d}\alpha = \sum_{p\leqslant n}(\log p)^2 \ll n\log n.$$

Next, we need the following analogous version of Weyl's inequality.

THEOREM 2.2. *Suppose that $a, q \in \mathbb{N}$ satisfy $(a, q) = 1$ and $q \leqslant n$. Suppose further that $\alpha \in \mathbb{R}$ satisfies $|\alpha - a/q| \leqslant q^{-2}$. Then*

$$|f(\alpha)| \ll (\log n)^4 (nq^{-\frac{1}{2}} + n^{\frac{4}{5}} + n^{\frac{1}{2}}q^{\frac{1}{2}}).$$

Using Dirichlet's theorem with $X = nP^{-1}$, we conclude that there exist $a, q \in \mathbb{Z}$ satisfying $(a, q) = 1$ and $1 \leqslant q \leqslant nP^{-1}$ such that

$$\left|\alpha - \frac{a}{q}\right| \leqslant q^{-1}Pn^{-1}.$$

Since $\alpha \in \mathfrak{m} \subseteq (Pn^{-1}, 1 - Pn^{-1})$, it follows that $1 \leqslant a \leqslant q$. Furthermore, we must have $q > P$, for otherwise $\alpha \in \mathfrak{M}$. It now follows from Theorem 2.2 that for every $\alpha \in \mathfrak{m}$, we must have

$$|f(\alpha)| \ll (\log n)^4(nq^{-\frac{1}{2}} + n^{\frac{4}{5}} + n^{\frac{1}{2}}q^{\frac{1}{2}}) \ll (\log n)^4(nP^{-\frac{1}{2}} + n^{\frac{4}{5}} + nP^{-\frac{1}{2}}) \tag{2.8}$$

$$\ll n(\log n)^{4-\frac{1}{2}B} \ll n(\log n)^{-2},$$

provided that $B \geqslant 12$. Combining (2.6)–(2.8), we conclude that

$$\left|\int_{\mathfrak{m}} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha\right| \ll n^2(\log n)^{-1}, \tag{2.9}$$

provided that $B \geqslant 12$.

We prove Theorem 2.2 in Section 2.5.

2.3. The Major Arcs

In this section, we study the integral

$$\int_{\mathfrak{M}(q,a)} f^3(\alpha)e(-\alpha n)\,\mathrm{d}\alpha,$$

where $a, q \in \mathbb{N}$ satisfy $1 \leqslant a \leqslant q \leqslant P$ and $(a, q) = 1$.

The first step in our argument is to find a suitable approximation to the generating function $f(\alpha)$. We introduce the function

$$v(\beta) = \sum_{m=1}^{n} e(\beta m).$$

THEOREM 2.3. *There is a positive constant C such that for every $a, q \in \mathbb{N}$ satisfying $1 \leqslant a \leqslant q \leqslant P$ and $(a, q) = 1$, and for any $\alpha \in \mathfrak{M}(q, a)$, we have*

$$f(\alpha) = \frac{\mu(q)}{\phi(q)} v\left(\alpha - \frac{a}{q}\right) + O(n\exp(-C(\log n)^{\frac{1}{2}})),$$

where $\mu : \mathbb{N} \to \mathbb{R}$ denotes the Möbius function.

The proof of Theorem 2.3 requires knowledge of the theory of the distribution of primes and the distribution of primes in arithmetic progressions, and will be given in Section 2.4.

It now follows from Theorem 2.3 that if $\alpha \in \mathfrak{M}(q, a)$, then

$$f^3(\alpha) - \frac{\mu(q)}{\phi^3(q)}v^3\left(\alpha - \frac{a}{q}\right) \ll n^2\left|f(\alpha) - \frac{\mu(q)}{\phi(q)}v\left(\alpha - \frac{a}{q}\right)\right| \ll n^3\exp(-C(\log n)^{\frac{1}{2}}).$$

Summing this error over all the major arcs, we obtain

$$\sum_{q\leqslant P}\sum_{\substack{a=1\\(a,q)=1}}^{q}\int_{\mathfrak{M}(q,a)}\left|f^3(\alpha) - \frac{\mu(q)}{\phi^3(q)}v^3\left(\alpha - \frac{a}{q}\right)\right|\mathrm{d}\alpha \ll P^3n^2\exp(-C(\log n)^{\frac{1}{2}}).$$

If we now write

$$R_1(n) = \sum_{q \leqslant P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}(q,a)} \frac{\mu(q)}{\phi^3(q)} v^3\left(\alpha - \frac{a}{q}\right) e(-\alpha n)\,\mathrm{d}\alpha,$$

then

$$\int_{\mathfrak{M}} f^3(\alpha) e(-\alpha n)\,\mathrm{d}\alpha = R_1(n) + O(P^3 n^2 \exp(-C(\log n)^{\frac{1}{2}})). \tag{2.10}$$

On the other hand,

$$\begin{aligned} R_1(n) &= \sum_{q \leqslant P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \int_{\mathfrak{M}(q,a)} \frac{\mu(q)}{\phi^3(q)} v^3\left(\alpha - \frac{a}{q}\right) e\left(-\left(\alpha - \frac{a}{q}\right) n\right) e\left(-\frac{an}{q}\right) \mathrm{d}\alpha \\ &= \sum_{q \leqslant P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi^3(q)} e\left(-\frac{an}{q}\right) \int_{\mathfrak{M}(q,a)} v^3\left(\alpha - \frac{a}{q}\right) e\left(-\left(\alpha - \frac{a}{q}\right) n\right) \mathrm{d}\alpha \\ &= \sum_{q \leqslant P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi^3(q)} e\left(-\frac{an}{q}\right) \int_{-Pn^{-1}}^{Pn^{-1}} v^3(\beta) e(-\beta n)\,\mathrm{d}\beta \\ &= \mathfrak{S}(n,P) J^*(n), \end{aligned} \tag{2.11}$$

where

$$\mathfrak{S}(n,P) = \sum_{q \leqslant P} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi^3(q)} e\left(-\frac{an}{q}\right)$$

and

$$J^*(n) = \int_{-Pn^{-1}}^{Pn^{-1}} v^3(\beta) e(-\beta n)\,\mathrm{d}\beta.$$

Our next task is to complete the series to infinity and to replace the interval of integration by a unit interval.

Let us first of all consider the integral $J^*(n)$. Since $v(\beta)$ is a geometric series, we clearly have

$$v(\beta) \ll \min\{n, \|\beta\|^{-1}\}.$$

It follows that if we write

$$J(n) = \int_{-\frac{1}{2}}^{\frac{1}{2}} v^3(\beta) e(-\beta n)\,\mathrm{d}\beta,$$

then

$$J(n) - J^*(n) \ll \int_{Pn^{-1}}^{\frac{1}{2}} \beta^{-3}\,\mathrm{d}\beta \ll P^{-2} n^2,$$

so that

$$\mathfrak{S}(n,P)(J(n) - J^*(n)) \ll P^{-2} n^2 \sum_{q \leqslant P} \frac{1}{\phi^2(q)} \ll P^{-1} n^2. \tag{2.12}$$

Combining (2.11) and (2.12), we have

$$R_1(n) = \mathfrak{S}(n,P) J(n) + o(n^2). \tag{2.13}$$

Note also that

$$J(n) = \sum_{m_1=1}^{n} \sum_{m_2=1}^{n} \sum_{m_3=1}^{n} \int_0^1 e(\beta(m_1 + m_2 + m_3 - n))\,\mathrm{d}\beta$$

is the number of solutions of the equation

$$m_1 + m_2 + m_3 = n, \quad 1 \leqslant m_1, m_2, m_3 \leqslant n,$$

so that

$$J(n) = \frac{(n-1)(n-2)}{2}. \tag{2.14}$$

Next, we consider the series $\mathfrak{S}(n, P)$. Write

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} \frac{\mu(q)}{\phi^3(q)} e\left(-\frac{an}{q}\right).$$

Then

$$\mathfrak{S}(n, P) = \mathfrak{S}(n) + O\left(\sum_{q>P} \frac{1}{\phi^2(q)}\right).$$

It is well known that $\phi(q) \gg q^{1-\delta}$ for any $\delta > 0$, so that

$$\sum_{q>P} \frac{1}{\phi^2(q)} \ll \sum_{q>P} \frac{1}{q^{\frac{3}{2}}} \ll P^{-\frac{1}{2}}.$$

In view of (2.14), we have

$$(\mathfrak{S}(n, P) - \mathfrak{S}(n))J(n) \ll n^2 P^{-\frac{1}{2}}. \tag{2.15}$$

Combining (2.13) and (2.15), we have

$$R_1(n) = \mathfrak{S}(n)J(n) + o(n^2). \tag{2.16}$$

Finally, we combine (2.3), (2.9), (2.10) and (2.16) to obtain

$$R(n) = \mathfrak{S}(n)J(n) + o(n^2).$$

Theorem 2.1 will follow if we can show that $\mathfrak{S}(n) \gg 1$. In fact, this follows from the result below.

THEOREM 2.4. *For every natural number $n \in \mathbb{N}$ satisfying $n > 1$, we have*

$$\mathfrak{S}(n) = \left(\prod_{p \nmid n} \left(1 + \frac{1}{(p-1)^3}\right)\right) \left(\prod_{p \mid n} \left(1 - \frac{1}{(p-1)^2}\right)\right).$$

REMARK. Note that $\mathfrak{S}(n) = 0$ when n is even. This is the only part in our discussion where the argument does not work when n is even.

We sketch the proof of Theorem 2.4 here. The details are left as an exercise.

SKETCH OF PROOF OF THEOREM 2.4. Clearly

$$\mathfrak{S}(n) = \sum_{q=1}^{\infty} \frac{\mu(q)}{\phi^3(q)} \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right) = \sum_{q=1}^{\infty} \frac{\mu(q)S(q)}{\phi^3(q)},$$

where

$$S(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} e\left(-\frac{an}{q}\right).$$

It can be shown that the function $S(q)$ is multiplicative, so that $S(qr) = S(q)S(r)$ whenever $(q, r) = 1$. On the other hand, recall that μ and ϕ are multiplicative. It follows that

$$\mathfrak{S}(n) = \prod_p \left(1 + \frac{\mu(p)S(p)}{\phi^3(p)} + \frac{\mu(p^2)S(p^2)}{\phi^3(p^2)} + \dots\right) = \prod_p \left(1 - \frac{S(p)}{\phi^3(p)}\right).$$

Note now that

$$\phi(p) = p - 1 \quad \text{and} \quad S(p) = \sum_{a=1}^{p-1} e\left(-\frac{an}{p}\right) = \begin{cases} p-1, & \text{if } p \mid n, \\ -1, & \text{if } p \nmid n. \end{cases}$$

The result follows immediately. ○

2.4. Input from the Distribution of Primes

Throughout this section, the letter C will denote a positive constant, suitably chosen so that the inequalities in question will hold. It may vary in value from one occurrence to the next.

Suppose that $y \geqslant 1$ is a given real number. Consider the sum

$$\sum_{p\leqslant y}(\log p)e\left(\frac{ap}{q}\right) = \sum_{r=1}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p\leqslant y\\ p\equiv r \bmod q}} \log p \tag{2.17}$$

$$= \sum_{\substack{r=1\\ (r,q)=1}}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p\leqslant y\\ p\equiv r \bmod q}} \log p + \sum_{\substack{r=1\\ (r,q)>1}}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p\leqslant y\\ p\equiv r \bmod q}} \log p.$$

If $(r,q) > 1$, then the only possibility of $p \equiv r \bmod q$ is that $r = (r,q)$ is a prime, and so

$$\sum_{\substack{p\leqslant y\\ p\equiv r \bmod q}} \log p = \begin{cases} \log r, & \text{if } r \leqslant y \text{ is a prime and divides } q,\\ 0, & \text{otherwise.}\end{cases}$$

It follows that

$$\sum_{\substack{r=1\\ (r,q)>1}}^{q} e\left(\frac{ar}{q}\right) \sum_{\substack{p\leqslant y\\ p\equiv r \bmod q}} \log p \ll \sum_{p\mid q} \log p \leqslant \log q. \tag{2.18}$$

To study the sum

$$\sum_{\substack{p\leqslant y\\ p\equiv r \bmod q}} \log p$$

when $(r,q) = 1$, recall that

$$\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{\chi(n)}{\chi(r)} = \begin{cases} 1, & \text{if } n \equiv r \bmod q,\\ 0, & \text{if } n \not\equiv r \bmod q.\end{cases}$$

Then

$$\sum_{\substack{p\leqslant y\\ p\equiv r \bmod q}} \log p = \sum_{p\leqslant y} \left(\frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{\chi(n)}{\chi(r)}\right) \log p = \frac{1}{\phi(q)} \sum_{\chi \bmod q} \frac{1}{\chi(r)} \sum_{p\leqslant y} \chi(p) \log p. \tag{2.19}$$

For the principal character χ_0 modulo q, it can be shown that

$$\left|\sum_{p\leqslant y} \chi_0(p) \log p - y\right| \ll y \exp(-C(\log y)^{\frac{1}{2}}) + \log q. \tag{2.20}$$

Similarly, it can be shown that for any non-principal character χ modulo q, we have

$$\left|\sum_{p\leqslant y} \chi(p) \log p\right| \ll y \exp(-C(\log y)^{\frac{1}{2}}). \tag{2.21}$$

REMARK. We do not prove (2.20) and (2.21) which are related to the Prime number theorem. However, it is a good reason for the choice of our generating function $f(\alpha)$. In the study of the distribution of primes, an inequality of the type

$$\left|\sum_{n\leqslant y} \Lambda(n) - y\right| \ll y \exp(-C(\log y)^{\frac{1}{2}})$$

will imply the Prime number theorem. Now it is easy to show that

$$0 \leqslant \sum_{n\leqslant y} \Lambda(n) - \sum_{p\leqslant y} \log p \leqslant y^{1/2} \log y,$$

so that

$$\left|\sum_{p\leqslant y}\log p - y\right| \ll y\exp(-C(\log y)^{\frac{1}{2}}).$$

On the other hand, we have

$$0 \leqslant \sum_{p\leqslant y}\log p - \sum_{p\leqslant y}\chi_0(p)\log p = \sum_{\substack{p\leqslant y\\(p,q)>1}}\log p \leqslant \sum_{p|q}\log p \leqslant \log q.$$

The inequality (2.20) follows.

Recall that on the major arc $\mathfrak{M}(q,a)$, we have $q \leqslant P = (\log n)^B$. If $n^{\frac{1}{2}} < y \leqslant n$, then it follows from (2.17)–(2.21) that

$$\sum_{p\leqslant y}(\log p)e\left(\frac{ap}{q}\right) = \frac{y}{\phi(q)}\sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{ar}{q}\right) + O(n\exp(-C(\log n)^{\frac{1}{2}}).$$

This inequality is trivial if $y \leqslant n^{\frac{1}{2}}$, and so holds for every y satisfying $1 \leqslant y \leqslant n$. We can also show that

$$\sum_{\substack{r=1\\(r,q)=1}}^{q} e\left(\frac{ar}{q}\right) = \mu(q),$$

so that

$$\sum_{p\leqslant y}(\log p)e\left(\frac{ap}{q}\right) - \frac{\mu(q)}{\phi(q)}y = O(n\exp(-C(\log n)^{\frac{1}{2}}). \tag{2.22}$$

We now use the partial summation formula (1.12) with $X = n$, $F(m) = e(\beta m)$ and

$$a_m = \begin{cases} (\log m)e\left(\dfrac{am}{q}\right) - \dfrac{\mu(q)}{\phi(q)}, & \text{if } m \text{ is a prime},\\ -\dfrac{\mu(q)}{\phi(q)}, & \text{if } m \text{ is not a prime}, \end{cases}$$

to obtain

$$\begin{aligned}&\sum_{p\leqslant n}(\log p)e\left(\frac{ap}{q}\right)e(\beta p) - \sum_{m=1}^{n}\frac{\mu(q)}{\phi(q)}e(\beta m)\\ &\quad= e(\beta n)\left(\sum_{p\leqslant n}(\log p)e\left(\frac{ap}{q}\right) - \sum_{m=1}^{n}\frac{\mu(q)}{\phi(q)}\right) - 2\pi\mathrm{i}\beta\int_0^n e(\beta y)\left(\sum_{p\leqslant y}(\log p)e\left(\frac{ap}{q}\right) - \sum_{m\leqslant y}\frac{\mu(q)}{\phi(q)}\right)\mathrm{d}y.\end{aligned}$$

It follows from (2.22) that

$$f(\alpha) - \frac{\mu(q)}{\phi(q)}v\left(\alpha - \frac{a}{q}\right) = O\left(\left(1 + n\left|\alpha - \frac{a}{q}\right|\right)n\exp(-C(\log n)^{\frac{1}{2}}\right).$$

This gives Theorem 2.3, in view of (2.4) and (2.5).

2.5. A Fundamental Identity of Vaughan

Suppose that $F : \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ is a given function. Suppose also that the variables x, y, z, d are natural numbers. Then for any real number $X > 1$, the identity

$$F(1,y) = \sum_{d\leqslant X}\sum_{z}\mu(d)F(zd,y) - \sum_{x>X}\left(\sum_{\substack{d|x\\d\leqslant X}}\mu(d)\right)F(x,y) \tag{2.23}$$

always holds as a consequence of the simple identity

$$\sum_{\substack{d|x\\ d\leqslant X}} \mu(d) = 0, \quad 1 < x \leqslant X,$$

and on noting that, writing $x = zd$, we have

$$\sum_{d\leqslant X}\sum_{z} \mu(d)F(zd,y) = \sum_{x}\left(\sum_{\substack{d,z\\ zd=x\\ d\leqslant X}} \mu(d)\right)F(x,y) = \sum_{x}\left(\sum_{\substack{d|x\\ d\leqslant X}} \mu(d)\right)F(x,y).$$

Suppose that the real parameter X satisfies $1 < X < n$. Writing

$$F(x,y) = \begin{cases} \Lambda(y)e(\alpha xy), & \text{if } X < y \leqslant n/x,\\ 0, & \text{otherwise,}\end{cases}$$

and summing (2.23) over y, we have

$$\begin{aligned}\sum_{X<y\leqslant n} \Lambda(y)e(\alpha y) &= \sum_{d\leqslant X}\sum_{z}\sum_{\substack{y>X\\ zdy\leqslant n}} \mu(d)\Lambda(y)e(\alpha zdy) - \sum_{x>X}\sum_{\substack{y>X\\ xy\leqslant n}}\left(\sum_{\substack{d|x\\ d\leqslant X}} \mu(d)\right)\Lambda(y)e(\alpha xy)\\ &= S_1 - S_2 - S_3,\end{aligned}$$

where

$$\begin{aligned} S_1 &= \sum_{d\leqslant X}\sum_{z}\sum_{\substack{y\\ zdy\leqslant n}} \mu(d)\Lambda(y)e(\alpha zdy),\\ S_2 &= \sum_{d\leqslant X}\sum_{z}\sum_{\substack{y\leqslant X\\ zdy\leqslant n}} \mu(d)\Lambda(y)e(\alpha zdy),\\ S_3 &= \sum_{x>X}\sum_{\substack{y>X\\ xy\leqslant n}}\left(\sum_{\substack{d|x\\ d\leqslant X}} \mu(d)\right)\Lambda(y)e(\alpha xy).\end{aligned}$$

First of all, writing $zy = x$, we have

$$\begin{aligned} S_1 &= \sum_{d\leqslant X}\sum_{x\leqslant n/d} \mu(d)e(\alpha dx)\left(\sum_{y|x}\Lambda(y)\right) = \sum_{d\leqslant X}\sum_{x\leqslant n/d} \mu(d)(\log x)e(\alpha dx)\\ &= \sum_{d\leqslant X}\mu(d)\sum_{x\leqslant n/d}\int_1^x e(\alpha dx)\frac{\mathrm{d}y}{y} = \sum_{d\leqslant X}\mu(d)\int_1^{n/d}\sum_{y\leqslant x\leqslant n/d} e(\alpha dx)\frac{\mathrm{d}y}{y}\\ &\ll \sum_{d\leqslant X}\left(\log\frac{n}{d}\right)\min\left\{\frac{n}{d}, \|\alpha d\|^{-1}\right\} \leqslant (\log n)\sum_{d\leqslant X}\min\left\{\frac{n}{d}, \|\alpha d\|^{-1}\right\}.\end{aligned}$$

Secondly, writing $dy = x$, we have

$$\begin{aligned} S_2 &= \sum_{x\leqslant X^2}\left(\sum_{d\leqslant X}\sum_{\substack{y\leqslant X\\ dy=x}} \mu(d)\Lambda(y)\right)\sum_{z\leqslant n/x} e(\alpha xz) \ll \sum_{x\leqslant X^2}\left(\sum_{y|x}\Lambda(y)\right)\left|\sum_{z\leqslant n/x} e(\alpha xz)\right|\\ &\ll \sum_{x\leqslant X^2}(\log x)\min\left\{\frac{n}{x}, \|\alpha x\|^{-1}\right\} \ll (\log X^2)\sum_{x\leqslant X^2}\min\left\{\frac{n}{x}, \|\alpha x\|^{-1}\right\}.\end{aligned}$$

We now make the choice

$$X = n^{\frac{2}{5}}. \tag{2.24}$$

Then it follows from Theorem 1.16 that

$$S_1, S_2 \ll (\log n)^2 (nq^{-1} + n^{\frac{4}{5}} + q).$$

To study the sum S_3, we partition the interval $(X, n/X)$. Let k_0 be the unique integer satisfying

$$2^{k_0} X^2 < n \leqslant 2^{k_0+1} X^2.$$

Then

$$\left(X, \frac{n}{X}\right) \subset \bigcup_{Y \in \mathcal{A}} (Y, 2Y], \tag{2.25}$$

where

$$\mathcal{A} = \{2^k X : k = 0, 1, \ldots, k_0\}, \tag{2.26}$$

and

$$S_3 = \sum_{Y \in \mathcal{A}} \Sigma(Y), \tag{2.27}$$

where, for every $Y \in \mathcal{A}$,

$$\Sigma(Y) = \sum_{Y < x \leqslant 2Y} \sum_{X < y \leqslant n/x} \left(\sum_{\substack{d|x \\ d \leqslant X}} \mu(d) \right) \Lambda(y) e(\alpha x y). \tag{2.28}$$

Note that while the two sides of (2.25) are not equal, note that $\Sigma(Y)$ in (2.28) has no contribution from any $x \geqslant n/X$, as the inner sum over y becomes an empty sum in this case.

For every $Y \in \mathcal{A}$, we have, by Cauchy's inequality,

$$\begin{aligned} |\Sigma(Y)|^2 &\leqslant \left(\sum_{Y < x \leqslant 2Y} \left| \sum_{\substack{d|x \\ d \leqslant X}} \mu(d) \right|^2 \right) \left(\sum_{Y < x \leqslant 2Y} \left| \sum_{X < y \leqslant n/x} \Lambda(y) e(\alpha x y) \right|^2 \right) \\ &\leqslant \left(\sum_{x \leqslant 2Y} d^2(x) \right) \left(\sum_{Y < x \leqslant 2Y} \left| \sum_{X < y \leqslant n/x} \Lambda(y) e(\alpha x y) \right|^2 \right), \end{aligned}$$

where $d : \mathbb{N} \to \mathbb{R}$ is the divisor function. It is well known that

$$\sum_{x \leqslant Z} d^2(x) \ll Z (\log 2Z)^3.$$

On the other hand,

$$\begin{aligned} \sum_{Y < x \leqslant 2Y} \left| \sum_{X < y \leqslant n/x} \Lambda(y) e(\alpha x y) \right|^2 &= \sum_{Y < x \leqslant 2Y} \sum_{X < y \leqslant n/x} \sum_{X < z \leqslant n/x} \Lambda(y) \Lambda(z) e(\alpha x (y - z)) \\ &= \sum_{y < n/Y} \sum_{z < n/Y} \Lambda(y) \Lambda(z) \sum_{\substack{Y < x \leqslant 2Y \\ x \leqslant n/y \\ x \leqslant n/z}} e(\alpha x (y - z)) \\ &\ll (\log n)^2 \sum_{y < n/Y} \sum_{z < n/Y} \min\{Y, \|\alpha(y - z)\|^{-1}\}. \end{aligned}$$

It follows that

$$|\Sigma(Y)|^2 \ll Y (\log n)^5 \sum_{y < n/Y} \sum_{z < n/Y} \min\{Y, \|\alpha(y - z)\|^{-1}\}. \tag{2.29}$$

To estimate the double sum in (2.29), we write $h = |y - z|$. Then

$$\sum_{y<n/Y} \sum_{z<n/Y} \min\{Y, \|\alpha(y-z)\|^{-1}\} \ll \sum_{y<n/Y} \sum_{0\leqslant h<n/Y} \min\{Y, \|\alpha h\|^{-1}\} \tag{2.30}$$
$$\ll n + \frac{n}{Y} \sum_{1\leqslant h<n/Y} \min\{Y, \|\alpha h\|^{-1}\}$$
$$\ll n + \frac{n}{Y} \sum_{1\leqslant h<n/Y} \min\{nh^{-1}, \|\alpha h\|^{-1}\}.$$

Applying Theorem 1.16 with $n = XY$, we deduce that if $q \leqslant n$, then

$$\sum_{1\leqslant h<n/Y} \min\{nh^{-1}, \|\alpha h\|^{-1}\} \ll n\left(\frac{1}{q} + \frac{1}{Y} + \frac{q}{n}\right)\log n. \tag{2.31}$$

Combining (2.29)–(2.31), we conclude that if $q \leqslant n$, then

$$|\Sigma(Y)|^2 \ll nY(\log n)^5 + n(nq^{-1} + nY^{-1} + q)(\log n)^6 \tag{2.32}$$
$$\ll n(\log n)^6(nq^{-1} + Y + nY^{-1} + q).$$

It now follows from (2.27) and (2.32) that

$$S_3 \ll \sum_{Y\in\mathcal{A}} (\log n)^3(nq^{-\frac{1}{2}} + n^{\frac{1}{2}}Y^{\frac{1}{2}} + nY^{-\frac{1}{2}} + n^{\frac{1}{2}}q^{\frac{1}{2}}) \ll (\log n)^4(nq^{-\frac{1}{2}} + n^{\frac{4}{5}} + n^{\frac{1}{2}}q^{\frac{1}{2}}),$$

in view of (2.24) and (2.26).

This completes the proof of Theorem 2.2.