CHAPTER 3

Diophantine Inequalities

3.1. Introduction

The Hardy–Littlewood method can be adapted to study solubility of equations of the form

$$c_1x_1^k + \ldots + c_sx_s^k = 0, \tag{3.1}$$

with fixed non-zero integer coefficients $c_1, \ldots, c_s$, not all having the same sign. Here the question is whether there are non-negative integers $x_1, \ldots, x_s$, not all zero, such that (3.1) holds.

A problem arises when the coefficients $c_1, \ldots, c_s$ are not all in rational ratio. Clearly, the trivial solution is the only solution, and it is inappropriate to study the equation (3.1) further. However, it is reasonable to ask whether the form $c_1x_1^k + \ldots + c_sx_s^k$ takes arbitrarily small values.

In this chapter, we discuss a variation of the Hardy–Littlewood method due to Davenport and Heilbronn in the 1940's and prove the following result.

THEOREM 3.1. *Suppose that $k \geqslant 2$ is a fixed integer, and the integer $s \geqslant 2^k + 1$. Suppose further that $\lambda_1, \ldots, \lambda_s$ are fixed non-zero real numbers, not all in rational ratio and not all having the same sign. Then for every positive real number η, there exist integers $x_1, \ldots, x_s$, not all zero, such that $|\lambda_1x_1^k + \ldots, \lambda_sx_s^k| < \eta$.*

Analogous to this, and also analogous to the ternary Goldbach problem, is the following result. The proof is left to the reader.

THEOREM 3.2. *Suppose that $\lambda_1, \lambda_2, \lambda_3$ are fixed non-zero real numbers, not all in rational ratio and not all having the same sign. Then for every positive real number η, there exist primes p_1, p_2, p_3 such that $|\lambda_1p_1 + \lambda_2p_2 + \lambda_3p_3| < \eta$.*

Note first of all that it suffices to prove Theorem 3.1 in the special case $\eta = 1$, for we can replace the coefficients λ_j by λ_j/η. Also, by relabelling if necessary, we may assume that λ_1/λ_2 is irrational. On the other hand, if $\lambda_1/\lambda_2 > 0$, then there exists $j > 2$ such that $\lambda_1/\lambda_j < 0$. If λ_1/λ_j is rational, then λ_2/λ_j is irrational and negative. Hence, by further relabelling if necessary, we may assume that

$$0 > \lambda_1/\lambda_2 \notin \mathbb{Q}.$$

We need a function that will pick out all real numbers β such that $|\beta| < 1$. Clearly, the identity

$$\int_{-\infty}^{\infty} e(\alpha\beta)\frac{\sin 2\pi\alpha}{\pi\alpha}\,\mathrm{d}\alpha = \begin{cases} 1, & \text{if } |\beta| < 1, \\ 0, & \text{if } |\beta| > 1, \end{cases}$$

appears to be ideal for this purpose. Unfortunately, there are difficulties associated with the use of this identity because the integral does not converge absolutely. Instead, we write

$$K(\alpha) = \left(\frac{\sin \pi\alpha}{\pi\alpha}\right)^2,$$

and use the identity

$$\int_{-\infty}^{\infty} e(\alpha\beta)K(\alpha)\,\mathrm{d}\alpha = \max\{1 - |\beta|, 0\}, \tag{3.2}$$

which can be established by using the Cauchy integral formula.

Suppose that N is a sufficiently large natural number, to be specified later. For every $j = 1, \ldots, s$, write

$$f_j(\alpha) = \sum_{x=1}^{N} e(\alpha\lambda_j x^k),$$

and consider

$$\begin{aligned} R(N) &= \int_{-\infty}^{\infty} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha)\,\mathrm{d}\alpha = \int_{-\infty}^{\infty} \sum_{x_1=1}^{N} \ldots \sum_{x_s=1}^{N} e(\alpha(\lambda_1 x_1^k + \ldots + \lambda_s x_s^k))K(\alpha)\,\mathrm{d}\alpha \\ &= \sum_{x_1=1}^{N} \ldots \sum_{x_s=1}^{N} \int_{-\infty}^{\infty} e(\alpha(\lambda_1 x_1^k + \ldots + \lambda_s x_s^k))K(\alpha)\,\mathrm{d}\alpha \\ &= \sum_{x_1=1}^{N} \ldots \sum_{x_s=1}^{N} \max\{1 - |\lambda_1 x_1^k + \ldots + \lambda_s x_s^k|, 0\}, \end{aligned}$$

in view of (3.2).

The size of the product

$$\prod_{j=1}^{s} f_j(\alpha) \tag{3.3}$$

varies greatly as the value of α varies. Roughly speaking, the product (3.3) is relatively large in size when α is close to the origin. However, the irrationality of λ_1/λ_2 ensures that one of $f_1(\alpha)$ or $f_2(\alpha)$ is relatively small when α is not near the origin. Hence there is only one major arc. On the other hand, the contribution for large α can be handled in a trivial manner. The idea of Davenport and Heilbronn is therefore to write

$$R(N) = \int_{\mathfrak{M}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha)\,\mathrm{d}\alpha + \int_{\mathfrak{m}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha)\,\mathrm{d}\alpha + \int_{\mathfrak{t}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha)\,\mathrm{d}\alpha,$$

where the three sets $\mathfrak{M}$, $\mathfrak{m}$ and $\mathfrak{t}$ form a partition of the set $\mathbb{R}$ of all real numbers.

Let ν be a sufficiently small positive real number, and write

$$P = N^{\nu}.$$

We consider the major arc

$$\mathfrak{M} = \{\alpha \in \mathbb{R} : |\alpha| \leqslant PN^{-k}\}, \tag{3.4}$$

two minor arcs

$$\mathfrak{m} = \{\alpha \in \mathbb{R} : PN^{-k} < |\alpha| \leqslant P\},$$

and the trivial regions

$$\mathfrak{t} = \{\alpha \in \mathbb{R} : |\alpha| > P\}.$$

To prove Theorem 3.1, it suffices to show that $R(N) > 0$ for all some $N \in \mathbb{N}$. Our strategy is to find some sufficiently small positive value of ν such that

$$\int_{\mathfrak{M}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha)\,\mathrm{d}\alpha \gg N^{s-k}$$

for all sufficiently large $N \in \mathbb{N}$, and such that there exist arbitrarily large values of $N \in \mathbb{N}$ for which

$$\int_{\mathfrak{m}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha)\,\mathrm{d}\alpha = o(N^{s-k}).$$

We also show that

$$\int_{\mathfrak{t}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha) \, \mathrm{d}\alpha = o(N^{s-k}).$$

Here, and in all subsequent argument, all implicit constants depend at most on k, s, ϵ and the coefficients $\lambda_1, \ldots, \lambda_s$.

3.2. The Trivial Regions

In this section, we study the integral

$$\int_{\mathfrak{t}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha) \, \mathrm{d}\alpha.$$

For every real number X and every $j = 1, \ldots, s$, we have

$$\int_X^{X+1} |f_j(\alpha)|^{2^k} \, \mathrm{d}\alpha \ll N^{2^k - k + \epsilon}$$

by Hua's lemma. It follows from Hölder's inequality that

$$\int_X^{X+1} \left| \prod_{j=1}^{2^k} f_j(\alpha) \right| \mathrm{d}\alpha \leqslant \prod_{j=1}^{2^k} \left(\int_X^{X+1} |f_j(\alpha)|^{2^k} \, \mathrm{d}\alpha \right)^{2^{-k}} \ll N^{2^k - k + \epsilon}. \tag{3.5}$$

Now

$$\begin{aligned}
\int_{\mathfrak{t}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha) \, \mathrm{d}\alpha &\ll \int_P^{\infty} \left| \prod_{j=1}^{s} f_j(\alpha) \right| \alpha^{-2} \, \mathrm{d}\alpha \leqslant \sum_{h=0}^{\infty} \int_{h+P}^{h+1+P} \left| \prod_{j=1}^{s} f_j(\alpha) \right| \alpha^{-2} \, \mathrm{d}\alpha \\
&\leqslant \sum_{h=0}^{\infty} \left(\sup_{\alpha \in [h+P, h+1+P]} \left| \prod_{j=2^k+1}^{s} f_j(\alpha) \right| \alpha^{-2} \right) \int_{h+P}^{h+1+P} \left| \prod_{j=1}^{2^k} f_j(\alpha) \right| \mathrm{d}\alpha \\
&\ll \sum_{h=0}^{\infty} N^{s-2^k} (h+P)^{-2} N^{2^k - k + \epsilon} = N^{s-k+\epsilon} \sum_{h=0}^{\infty} (h+P)^{-2} \\
&\ll N^{s-k+\epsilon} P^{-1} = o(N^{s-k}).
\end{aligned}$$

3.3. The Major Arc

In this section, we study the integral

$$\int_{\mathfrak{M}} \left(\prod_{j=1}^{s} f_j(\alpha) \right) K(\alpha) \, \mathrm{d}\alpha.$$

It is here that we use the condition that $\lambda_1/\lambda_2 < 0$.

The first step in our argument here is to find suitable approximations to the generating functions $f_j(\alpha)$. Our argument here is considerably simpler than that in Chapter 1, since there is only one major arc.

THEOREM 3.3. *Suppose that $\alpha \in \mathfrak{M}$, and that for every $j = 1, \ldots, s$,*

$$v_j(\alpha) = \int_0^N e(\alpha \lambda_j \beta^k) \, \mathrm{d}\beta.$$

Then

$$f_j(\alpha) - v_j(\alpha) = O(P).$$

Remark. Suppose that the function F has continuous derivative on the interval $[0,N]$. Then

$$\sum_{m\leqslant N} F(m)=\int_0^N F(y)\,\mathrm{d}y+\int_0^N F'(y)(y-[y])\,\mathrm{d}y.$$

This can be obtained from the formula (1.12) for partial summation with $X=N$ and $a_m=1$ and integration by parts.

Proof of Theorem 3.3. We take $F(y)=e(\alpha\lambda_j y^k)$. Then it follows from the Remark above that

$$f_j(\alpha)-v_j(\alpha)=2\pi\mathrm{i}\alpha\lambda_j\int_0^N ky^{k-1}e(\alpha\lambda_j y^k)(y-[y])\,\mathrm{d}y\ll|\alpha|\int_0^N ky^{k-1}\,\mathrm{d}y=N^k|\alpha|\leqslant P,$$

in view of (3.4). ○

It now follows from Theorem 3.3 that if $\alpha\in\mathfrak{M}$, then

$$\prod_{j=1}^s f_j(\alpha)-\prod_{j=1}^s v_j(\alpha)=\sum_{j=1}^s(f_j(\alpha)-v_j(\alpha))\left(\prod_{i<j}f_i(\alpha)\right)\left(\prod_{i>j}v_i(\alpha)\right)\ll N^{s-1}P.$$

Hence it follows from (3.4) and the observation $K(\alpha)\ll 1$ that

$$\int_{\mathfrak{M}}\left(\prod_{j=1}^s f_j(\alpha)-\prod_{j=1}^s v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha\ll N^{s-k-1}P^2=o(N^{s-k}),\tag{3.6}$$

if ν is sufficiently small.

We now study the integral

$$\int_{\mathfrak{M}}\left(\prod_{j=1}^s v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha.$$

Our next task is to replace the interval of integration by $(-\infty,\infty)$, noting that the functions $v_j(\alpha)$ decay rather rapidly away from the origin. More precisely, we have the following estimate.

Theorem 3.4. *Suppose that $\alpha\in\mathbb{R}$ is non-zero. Then for every $j=1,\dots,s$,*

$$v_j(\alpha)\ll|\alpha|^{-1/k}.$$

Proof. Using the substitution $\gamma=\alpha\lambda_j\beta^k$, we have

$$v_j(\alpha)=\frac{1}{(\alpha\lambda_j)^{1/k}}\int_0^{\alpha\lambda_j N^k}\frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma.$$

On the other hand,

$$\int_0^X\frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma\leqslant 2$$

for every $X\geqslant 0$. To see this, suppose first of all that $X\in[0,1]$. Then

$$\left|\int_0^X\frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma\right|\leqslant\int_0^1\frac{1}{k}\gamma^{1/k-1}\,\mathrm{d}\gamma=1.$$

Suppose now that $X>1$. Then

$$\left|\int_0^X\frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma\right|\leqslant\left|\int_0^1\frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma\right|+\left|\int_1^X\frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma\right|.$$

Clearly

$$\left|\int_0^1\frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma\right|\leqslant 1.$$

Furthermore, integrating by parts and using the Triangle inequality, we have

$$\left|\int_1^X \frac{1}{k}\gamma^{1/k-1}e(\gamma)\,\mathrm{d}\gamma\right| \leqslant \left|\frac{1}{2\pi\mathrm{i}k}(X^{1/k-1}e(X)-1)\right| + \left|\frac{1}{2\pi\mathrm{i}k}\int_1^X\left(\frac{1}{k}-1\right)\gamma^{1/k-2}e(\gamma)\,\mathrm{d}\gamma\right|$$
$$\leqslant \frac{1}{\pi k}+\frac{1}{2\pi k}\left|\int_1^X\left(\frac{1}{k}-1\right)\gamma^{1/k-2}e(\gamma)\,\mathrm{d}\gamma\right| \leqslant 1.$$

This completes the proof. ○

It now follows from Theorem 3.4 that

$$\int_{\mathbb{R}\setminus\mathfrak{M}}\left(\prod_{j=1}^s v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha \ll \int_{PN^{-k}}^\infty \alpha^{-s/k}\,\mathrm{d}\alpha \ll N^{s-k}P^{1-s/k}=o(N^{s-k}). \tag{3.7}$$

Combining (3.6) and (3.7), we conclude that

$$\int_{\mathfrak{M}}\left(\prod_{j=1}^s f_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha-\int_{-\infty}^\infty\left(\prod_{j=1}^s v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha=o(N^{s-k}).$$

On the other hand,

$$\int_{-\infty}^\infty\left(\prod_{j=1}^s v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha=\int_{-\infty}^\infty\left(\int_0^N\ldots\int_0^N e(\alpha(\lambda_1\beta_1^k+\ldots+\lambda_s\beta_s^k))\,\mathrm{d}\beta_1\ldots\mathrm{d}\beta_s\right)K(\alpha)\,\mathrm{d}\alpha.$$

Since $K(\alpha)\ll\min\{1,\alpha^{-2}\}$ and the integrand is continuous, we interchange the order of integration. Then

$$\begin{aligned}\int_{-\infty}^\infty\left(\prod_{j=1}^s v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha \\ &=\int_0^N\ldots\int_0^N\left(\int_{-\infty}^\infty e(\alpha(\lambda_1\beta_1^k+\ldots+\lambda_s\beta_s^k))K(\alpha)\,\mathrm{d}\alpha\right)\mathrm{d}\beta_1\ldots\mathrm{d}\beta_s\\ &=\int_0^N\ldots\int_0^N\max\{1-|\lambda_1\beta_1^k+\ldots+\lambda_s\beta_s^k|,0\}\,\mathrm{d}\beta_1\ldots\mathrm{d}\beta_s\\ &=\frac{1}{k^s}\int_0^{N^k}\ldots\int_0^{N^k}(\gamma_1\ldots\gamma_s)^{1/k-1}\max\{1-|\lambda_1\gamma_1+\ldots+\lambda_s\gamma_s|,0\}\,\mathrm{d}\gamma_1\ldots\mathrm{d}\gamma_s,\end{aligned} \tag{3.8}$$

using the substitution $\gamma_j=\beta_j^k$ for every $j=1,\ldots,s$.

Now we make use of our assumption that $\lambda_1/\lambda_2<0$. Note that the region

$$\mathcal{B}=\{(\gamma_2,\ldots,\gamma_s):\gamma_2\in[\delta N^k,2\delta N^k],\ \gamma_3,\ldots,\gamma_s\in[\delta^2N^k,2\delta^2N^k]\}$$

is contained in $[0,N^k]^{s-1}$ whenever $0\leqslant\delta\leqslant1/2$. Also, if δ is sufficiently small relative to $\lambda_1,\ldots,\lambda_s$, then for every $(\gamma_2,\ldots,\gamma_s)\in\mathcal{B}$, we have

$$-\frac{\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s}{\lambda_1}\geqslant\left|\frac{\lambda_2}{\lambda_1}\right|\delta N^k-\frac{|\lambda_3|+\ldots+|\lambda_s|}{|\lambda_1|}2\delta^2N^k>2\delta^2N^k$$

and

$$-\frac{\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s}{\lambda_1}\leqslant\left|\frac{\lambda_2}{\lambda_1}\right|2\delta N^k+\frac{|\lambda_3|+\ldots+|\lambda_s|}{|\lambda_1|}2\delta^2N^k<\frac{1}{2}N^k.$$

Note next that the condition

$$|\lambda_1\gamma_1+\ldots+\lambda_s\gamma_s|\leqslant\frac{1}{2} \tag{3.9}$$

is equivalent to the condition

$$-\frac{1}{2}-(\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s)\leqslant\lambda_1\gamma_1\leqslant\frac{1}{2}-(\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s);$$

in other words,

$$-\frac{1}{2|\lambda_1|}-\frac{\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s}{\lambda_1}\leqslant\gamma_1\leqslant\frac{1}{2|\lambda_1|}-\frac{\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s}{\lambda_1}.$$

It follows that if $(\gamma_2,\ldots,\gamma_s)\in\mathcal{B}$ and (3.9) holds, then

$$-\frac{1}{2|\lambda_1|}+2\delta^2N^k<\gamma_1<\frac{1}{2|\lambda_1|}+\frac{1}{2}N^k,$$

and so

$$\delta^2N^k<\gamma_1<N^k,$$

if N is sufficiently large.

We now note that the intrgrand on the right hand side of (3.8) is non-negative. Hence if we restrict the integration to those $(\gamma_1,\ldots,\gamma_s)$ satisfying $(\gamma_2,\ldots,\gamma_s)\in\mathcal{B}$ and the condition (3.9), then we have

$$\int_{-\infty}^{\infty}\left(\prod_{j=1}^{s}v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha\gg(N^{1-k})^s\int_{\mathcal{B}}\left(\int_{\mathcal{A}(\gamma_2,\ldots,\gamma_s)}\mathrm{d}\gamma_1\right)\mathrm{d}\gamma_2\ldots\mathrm{d}\gamma_s,$$

where the inner integral is over the interval

$$\mathcal{A}(\gamma_2,\ldots,\gamma_s)=\left[-\frac{1}{2|\lambda_1|}-\frac{\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s}{\lambda_1},\frac{1}{2|\lambda_1|}-\frac{\lambda_2\gamma_2+\ldots+\lambda_s\gamma_s}{\lambda_1}\right].$$

Since the volume of $\mathcal{B}$ is $\gg(N^k)^{s-1}$, it follows that

$$\int_{-\infty}^{\infty}\left(\prod_{j=1}^{s}v_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha\gg N^{(1-k)s}N^{k(s-1)}=N^{s-k},$$

if N is sufficiently large.

3.4. The Minor Arcs

In this section, we study the integral

$$\int_{\mathfrak{m}}\left(\prod_{j=1}^{s}f_j(\alpha)\right)K(\alpha)\,\mathrm{d}\alpha.$$

It is here that we use the condition that λ_1/λ_2 is irrational. It is also here that the argument requires a specialization of N.

Theorem 3.5. *Suppose that $a\in\mathbb{Z}$ and $q\in\mathbb{N}$ satisfy $(a,q)=1$ and*

$$\left|\frac{\lambda_1}{\lambda_2}-\frac{a}{q}\right|\leqslant\frac{1}{q^2}. \tag{3.10}$$

Suppose further that $N=q^2$. Then

$$\sup_{\alpha\in\mathfrak{m}}\min\{|f_1(\alpha)|,|f_2(\alpha)|\}\ll N^{1-\delta}$$

for some suitably small $\delta>0$.

Remark. The existence of arbitrarily large q, and hence N, satisfying (3.10) is guaranteed by Dirichlet's theorem. To see this, suppose on the contrary that there are only finitely many rational numbers a/q, with $a\in\mathbb{Z}$ and $q\in\mathbb{N}$, such that (3.10) holds. Let these be

$$\frac{a_1}{q_1},\ldots,\frac{a_\ell}{q_\ell}. \tag{3.11}$$

Since λ_1/λ_2 is irrational, $|\lambda_1/\lambda_2-a/q|>0$ for every $j=1,\ldots,\ell$. It follows that there exists $X\geqslant1$ such that

$$\left|\frac{\lambda_1}{\lambda_2}-\frac{a_j}{q_j}\right|>\frac{1}{X},\quad j=1,\ldots,\ell.$$

By Theorem 1.4, there exist $a, q \in \mathbb{Z}$ satisfying $1 \leqslant q \leqslant X$ and

$$\left|\frac{\lambda_1}{\lambda_2} - \frac{a}{q}\right| \leqslant \frac{1}{qX} \leqslant \frac{1}{q^2},$$

contradicting that (3.11) represents all the solutions of (3.10).

PROOF OF THEOREM 3.5. We assume that N is sufficiently large in relation to $\lambda_1, \ldots, \lambda_s$. Suppose that $\alpha \in \mathfrak{m}$. Write $X = N^k P^{-1/2}$. Applying Theorem 1.4, we see that for $i = 1, 2$, there exist $a_i, q_i \in \mathbb{Z}$ satisfying $(a_i, q_i) = 1$ and $1 \leqslant q_i \leqslant X$ such that

$$\left|\lambda_i \alpha - \frac{a_i}{q_i}\right| \leqslant \frac{1}{q_i X}. \tag{3.12}$$

We first of all establish that at least one of q_1, q_2 is relatively large. More precisely, we show that

$$\max\{q_1, q_2\} \gg N^{\frac{1}{5}}. \tag{3.13}$$

Suppose first of all that $a_i = 0$. Then

$$|\alpha| \leqslant \frac{1}{q_i X |\lambda_i|} \leqslant \frac{1}{X|\lambda_i|} = \frac{P^{\frac{1}{2}} N^{-k}}{|\lambda_i|} < PN^{-k},$$

if N is sufficiently large in relation to λ_i, so that $\alpha \in \mathfrak{M}$, a contradiction. Hence $a_i \neq 0$. On the other hand, it follows from (3.12) that

$$\lambda_i \alpha = \frac{a_i}{q_i} + \frac{\theta_i}{q_i X} = \frac{a_i}{q_i}\left(1 + \frac{\theta_i}{a_i X}\right)$$

for some $\theta_i \in \mathbb{R}$ satisfying $|\theta_i| \leqslant 1$. Hence

$$\frac{\lambda_1}{\lambda_2} = \frac{\lambda_1 \alpha}{\lambda_2 \alpha} = \frac{a_1 q_2}{a_2 q_1}\left(1 + \frac{\theta_1}{a_1 X}\right)\left(1 + \frac{\theta_2}{a_2 X}\right)^{-1}.$$

Since N is large, it follows that X is large, and so

$$\frac{1}{2}\left|\frac{\lambda_1}{\lambda_2}\right| < \left|\frac{a_1 q_2}{a_2 q_1}\right| < 2\left|\frac{\lambda_1}{\lambda_2}\right|,$$

whence

$$\frac{\lambda_1}{\lambda_2} = \frac{a_1 q_2}{a_2 q_1} + O\left(\left|\frac{a_1 q_2}{a_2 q_1}\right| X^{-1}\right) = \frac{a_1 q_2}{a_2 q_1} + O(X^{-1}),$$

where the implicit constants may depend on λ_1 and λ_2. Note next that it follows from (3.10) that

$$\frac{\lambda_1}{\lambda_2} = \frac{a}{q} + \frac{\theta}{q^2}$$

for some $\theta \in \mathbb{R}$ satisfying $|\theta| \leqslant 1$. Hence

$$\frac{a}{q} - \frac{a_1 q_2}{a_2 q_1} \ll X^{-1} + q^{-2} \ll N^{-1} = q^{-2}. \tag{3.14}$$

If the left hand side of (3.14) is non-zero, then

$$\frac{1}{|q a_2 q_1|} \leqslant \left|\frac{a}{q} - \frac{a_1 q_2}{a_2 q_1}\right| \ll q^{-2},$$

so that $|a_2 q_1| \gg q$. If the left hand side of (3.14) is zero, then since $(a, q) = 1$, it follows that $a_2 q_1$ must be an integer multiple of q, and so $|a_2 q_1| \gg q$ again. Also, $a_2 = \lambda_2 \alpha q_2 - \theta_2 X^{-1} \ll q_2 P$. It follows that $q_1 q_2 P \gg |a_2 q_1| \gg q$, so that $q_1 q_2 \gg q P^{-1} = N^{1/2} P^{-1}$, whence $\max\{q_1, q_2\} \gg N^{\frac{1}{4}} P^{-\frac{1}{2}} \gg N^{\frac{1}{5}}$ as required, if ν is sufficiently small. Next, by Weyl's inequality, given in Theorem 1.3, we have

$$f_i(\alpha) \ll N^{1+\epsilon}\left(\frac{1}{q_i} + \frac{1}{N} + \frac{q_i}{N^k}\right)^{2^{1-k}} \ll N^{1+\epsilon} q_i^{-2^{1-k}} + N^{1-\delta} \tag{3.15}$$

for some suitably small $\delta > 0$. Combining (3.13) and (3.15), we conclude that

$$\min\{|f_1(\alpha)|, |f_2(\alpha)|\} \ll N^{1-\delta}$$

for some suitably small $\delta > 0$. ○

To complete our discussion of the minor arcs, we assume that N is sufficiently large and chosen according to the specialization in Theorem 3.5. We also partition the minor arcs $\mathfrak{m}$ into

$$\mathfrak{m}_1 = \{\alpha \in \mathfrak{m} : |f_1(\alpha)| \leqslant |f_2(\alpha)|\} \quad \text{and} \quad \mathfrak{m}_2 = \mathfrak{m} \setminus \mathfrak{m}_1.$$

Analogous to (3.5), we can show that for $i = 1, 2$, we have, for every real number X,

$$\int_X^{X+1} \left| \prod_{\substack{j=1 \\ j \neq i}}^{2^k+1} f_j(\alpha) \right| \mathrm{d}\alpha \ll N^{2^k - k + \epsilon}.$$

Since $K(\alpha) \ll \min\{1, \alpha^{-2}\}$, it follows that

$$\int_{\mathfrak{m}} \left| \prod_{\substack{j=1 \\ j \neq i}}^{2^k+1} f_j(\alpha) \right| K(\alpha)\, \mathrm{d}\alpha \ll N^{2^k - k + \epsilon}.$$

It now follows from Theorem 3.5 that for $i = 1, 2$,

$$\int_{\mathfrak{m}_i} \left| \prod_{j=1}^{2^k+1} f_j(\alpha) \right| K(\alpha)\, \mathrm{d}\alpha \ll N^{2^k + 1 - k - \delta + \epsilon}.$$

Using the trivial bound $f_j(\alpha) \ll N$ for $j = 2^k + 2, \ldots, s$, we conclude that

$$\int_{\mathfrak{m}} \left| \prod_{j=1}^{s} f_j(\alpha) \right| K(\alpha)\, \mathrm{d}\alpha \ll N^{s - k - \delta + \epsilon} = o(N^{s-k}),$$

as required.