INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 1

THE REAL NUMBERS AND COUNTABILITY

1.1. Introduction

We shall only give a brief introduction of the basic properties of the real numbers, and denote the set of all real numbers by $\mathbb{R}$.

The first set of properties of $\mathbb{R}$ is generally known as the Field axioms. We offer no proof of these properties, and simply accept them as given.

FIELD AXIOMS.

(A1) *For every $a, b \in \mathbb{R}$, we have $a + b \in \mathbb{R}$.*
(A2) *For every $a, b, c \in \mathbb{R}$, we have $a + (b + c) = (a + b) + c$.*
(A3) *For every $a \in \mathbb{R}$, we have $a + 0 = a$.*
(A4) *For every $a \in \mathbb{R}$, there exists $-a \in \mathbb{R}$ such that $a + (-a) = 0$.*
(A5) *For every $a, b \in \mathbb{R}$, we have $a + b = b + a$.*
(M1) *For every $a, b \in \mathbb{R}$, we have $ab \in \mathbb{R}$.*
(M2) *For every $a, b, c \in \mathbb{R}$, we have $a(bc) = (ab)c$.*
(M3) *For every $a \in \mathbb{R}$, we have $a1 = a$.*
(M4) *For every $a \in \mathbb{R}$ such that $a \neq 0$, there exists $a^{-1} \in \mathbb{R}$ such that $aa^{-1} = 1$.*
(M5) *For every $a, b \in \mathbb{R}$, we have $ab = ba$.*
(D) *For every $a, b, c \in \mathbb{R}$, we have $a(b + c) = ab + ac$.*

Remark. The properties (A1)–(A5) concern the operation addition, while the properties (M1)–(M5) concern the operation multiplication. In the terminology of group theory, we say that the set $\mathbb{R}$ forms an abelian group under addition, and that the set of all non-zero real numbers forms an abelian group under multiplication. We also say that the set $\mathbb{R}$ forms a field under addition and multiplication.

The set of all real numbers also possesses an ordering relation, so we have the Order Axioms.

ORDER AXIOMS.
(O1) For every $a, b \in \mathbb{R}$, exactly one of $a < b$, $a = b$, $a > b$ holds.
(O2) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $b > c$, we have $a > c$.
(O3) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$, we have $a + c > b + c$.
(O4) For every $a, b, c \in \mathbb{R}$ satisfying $a > b$ and $c > 0$, we have $ac > bc$.

An important subset of the set $\mathbb{R}$ of all real numbers is the set

$$\mathbb{N} = \{1, 2, 3, \ldots\}$$

of all natural numbers. However, this definition does not bring out some of the main properties of the set $\mathbb{N}$ in a natural way. The following more complicated definition is therefore sometimes preferred.

Definition. The set $\mathbb{N}$ of all natural numbers is defined by the following four conditions:
(N1) $1 \in \mathbb{N}$.
(N2) If $n \in \mathbb{N}$, then the number $n + 1$, called the successor of n, also belongs to $\mathbb{N}$.
(N3) Every $n \in \mathbb{N}$ other than 1 is the successor of some number in $\mathbb{N}$.
(WO) Every non-empty subset of $\mathbb{N}$ has a least element.

Remark. The condition (WO) is called the Well-ordering principle.

To explain the significance of each of these four requirements, note that the conditions (N1) and (N2) together imply that $\mathbb{N}$ contains $1, 2, 3, \ldots$. However, these two conditions alone are insufficient to exclude from $\mathbb{N}$ numbers such as 5.5. Now, if $\mathbb{N}$ contained 5.5, then by condition (N3), $\mathbb{N}$ must also contain $4.5, 3.5, 2.5, 1.5, 0.5, -0.5, -1.5, -2.5, \ldots$, and so would not have a least element. We therefore exclude this possibility by stipulating that $\mathbb{N}$ has a least element. This is achieved by the condition (WO).

It can be shown that the condition (WO) implies the Principle of induction. The following two forms of the Principle of induction are particularly useful.

PRINCIPLE OF INDUCTION (WEAK FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIW1) $p(1)$ is true; and
(PIW2) $p(n + 1)$ is true whenever $p(n)$ is true.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

PRINCIPLE OF INDUCTION (STRONG FORM). *Suppose that the statement $p(.)$ satisfies the following conditions:*
(PIS1) $p(1)$ is true; and
(PIS2) $p(n + 1)$ is true whenever $p(m)$ is true for all $m \leq n$.
Then $p(n)$ is true for every $n \in \mathbb{N}$.

1.2. Completeness of the Real Numbers

The set $\mathbb{Z}$ of all integers is an extension of the set $\mathbb{N}$ of all natural numbers to include 0 and all numbers of the form $-n$, where $n \in \mathbb{N}$. The set $\mathbb{Q}$ of all rational numbers is the set of all real numbers of the form pq^{-1}, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$.

We see that the Field axioms and Order axioms hold good if the set $\mathbb{R}$ is replaced by the set $\mathbb{Q}$. On the other hand, the set $\mathbb{Q}$ is incomplete. A good illustration is the following result.

THEOREM 1A. *No rational number $x \in \mathbb{Q}$ satisfies $x^2 = 2$.*

PROOF. Suppose that pq^{-1} has square 2, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We may assume, without loss of generality, that p and q have no common factors apart from ± 1. Then $p^2 = 2q^2$ is even, so that p is even. We can write $p = 2r$, where $r \in \mathbb{Z}$. Then $q^2 = 2r^2$ is even, so that q is even, contradicting that assumption that p and q have no common factors apart from ± 1. ◯

It follows that the real number we know as $\sqrt{2}$ does not belong to $\mathbb{Q}$. We shall now discuss a property that distinguishes the set $\mathbb{R}$ from the set $\mathbb{Q}$.

DEFINITION. A non-empty set S of real numbers is said to be bounded above if there exists a number $K \in \mathbb{R}$ such that $x \leq K$ for every $x \in S$. The number K is called an upper bound of the set S. A non-empty set S of real numbers is said to be bounded below if there exists a number $k \in \mathbb{R}$ such that $x \geq k$ for every $x \in S$. The number k is called a lower bound of the set S. Furthermore, a non-empty set S of real numbers is said to be bounded if it is bounded above and below.

EXAMPLE 1.2.1. The set $\mathbb{N}$ is bounded below but not bounded above. See Section 1.3 for further discussion.

EXAMPLE 1.2.2. The set $\mathbb{Q}$ is neither bounded above nor bounded below.

EXAMPLE 1.2.3. The set $\{x \in \mathbb{R} : -1 < x < 1\}$ is bounded.

The axiom that distinguishes the set $\mathbb{R}$ from the set $\mathbb{Q}$ is the Completeness axiom. It can be stated in many equivalent forms. Here we state it as the Axiom of bound.

AXIOM OF BOUND (UPPER BOUND). *Suppose that S is a non-empty set of real numbers and S is bounded above. Then there is a number $M \in \mathbb{R}$ such that*
(B1) M is an upper bound of S; and
(B2) given any $\epsilon > 0$, there exists $s \in S$ such that $s > M - \epsilon$.

REMARK. Note that (B2) essentially says that any real number less than M cannot be an upper bound of S. In other words, M is the least upper bound of S. Note the important point here that the number M is a real number.

The axiom can be stated in the obvious alternative form below.

AXIOM OF BOUND (LOWER BOUND). *Suppose that S is a non-empty set of real numbers and S is bounded below. Then there is a number $m \in \mathbb{R}$ such that*
(b1) m is a lower bound of S; and
(b2) given any $\epsilon > 0$, there exists $s \in S$ such that $s < m + \epsilon$.

DEFINITION. The real number M satisfying (B1) and (B2) is called the supremum (or least upper bound) of S and denoted by $M = \sup S$. The real number m satisfying (b1) and (b2) is called the infimum (or greatest lower bound) of S and denoted by $m = \inf S$.

DEFINITION. Any number in $\mathbb{R} \setminus \mathbb{Q}$ is called an irrational number.

We now show that $\sqrt{2}$ is a real number.

THEOREM 1B. *There is a positive real number M satisfying $M^2 = 2$.*

PROOF. Let $S = \{x \in \mathbb{R} : x^2 < 2\}$. Since $0 \in S$, the set S is non-empty. On the other hand, it is easy to see that 2 is an upper bound of S; for if $x > 2$, then $x^2 > 4$. Hence S is bounded above. By the Axiom of bound, S has a supremum $M \in \mathbb{R}$. Clearly $M > 0$, since $1 \in S$. It remains to show that $M^2 = 2$.

Suppose on the contrary that $M^2 \neq 2$. Then by Axiom (O1), we must have $M^2 < 2$ or $M^2 > 2$. Suppose first of all that $M^2 < 2$. Then

$$(M+\epsilon)^2 = M^2 + 2M\epsilon + \epsilon^2 < M^2 + (2M+1)\epsilon < 2 \qquad \text{if} \qquad 0 < \epsilon < \min\left\{1, \frac{2-M^2}{2M+1}\right\},$$

contradicting that M is an upper bound of S. Suppose next that $M^2 > 2$, then

$$(M-\epsilon)^2 = M^2 - 2M\epsilon + \epsilon^2 > M^2 - 2M\epsilon > 2 \qquad \text{if} \qquad 0 < \epsilon < \frac{M^2-2}{2M},$$

contradicting that M is the least upper bound of S. We must therefore have $M^2 = 2$. ◯

REMARK. The above argument can be adapted to prove the following more general result: Suppose that $n \in \mathbb{N}$, $c \in \mathbb{R}$ and $c > 0$. Then the equation $x^n = c$ has a unique solution for $x \in \mathbb{R}$ and $x > 0$.

1.3. Consequences of the Completeness Axiom

In this section, we shall prove two simple consequences of the Completeness axiom. The first of these shows that there are arbitrarily large natural numbers, while the second shows that rational numbers and irrational numbers are everywhere along the real line.

THEOREM 1C. (ARCHIMEDEAN PROPERTY) *For every $x \in \mathbb{R}$, there exists $n \in \mathbb{N}$ such that $n > x$.*

PROOF. Suppose that $x \in \mathbb{R}$, and suppose on the contrary that $n \leq x$ for every $n \in \mathbb{N}$. Then x is an upper bound of $\mathbb{N}$, so that $\mathbb{N}$ is bounded above. By the Axiom of bound, the set $\mathbb{N}$ has a supremum, M say. Then

$$M \geq n \qquad \text{for every } n = 1, 2, 3, \ldots.$$

In particular, dropping the case $n = 1$, we have

$$M \geq n \qquad \text{for every } n = 2, 3, 4, \ldots.$$

Now every $n = 2, 3, 4, \ldots$ can be written as $k + 1$, where $k = 1, 2, 3, \ldots$ respectively. Hence

$$M \geq k + 1 \qquad \text{for every } k = 1, 2, 3, \ldots,$$

so that

$$M - 1 \geq k \qquad \text{for every } k = 1, 2, 3, \ldots;$$

in other words, $M - 1$ is an upper bound of $\mathbb{N}$. This contradicts the assumption that M is the supremum of $\mathbb{N}$. ◯

We are now in a position to prove the following important result.

THEOREM 1D. *The rational and irrational numbers are dense in $\mathbb{R}$. More precisely, between any two distinct real numbers, there exist a rational number and an irrational number.*

PROOF. Suppose that $x, y \in \mathbb{R}$ and $x < y$.

(a) We shall show that there exists $r \in \mathbb{Q}$ such that $x < r < y$. Suppose first of all that $x > 0$. By the Archimedean property, there exists $q \in \mathbb{N}$ such that $q > 1/(y - x)$, so that $q(y - x) > 1$. Consider the

positive real number qx. By the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > qx$. It follows that $S = \{n \in \mathbb{N} : n > qx\}$ is a non-empty set of natural numbers, and so has a least element p, in view of (WO). We now claim that $p - 1 \leq qx$. To see this, note that if $p = 1$, then $p - 1 = 0 < qx$. On the other hand, if $p \neq 1$, then $p - 1 > qx$ would contradict the definition of p. It now follows that

$$qx < p = (p-1) + 1 < qx + q(y-x) = qy,$$

so that

$$x < \frac{p}{q} < y.$$

Suppose now that $x \leq 0$. Then by the Archimedean property, there exists $k \in \mathbb{N}$ such that $k > -x$, so that $k + x > 0$. Then there exists $s \in \mathbb{Q}$ such that $x + k < s < y + k$, so that

$$x < s - k < y.$$

Clearly $r = s - k \in \mathbb{Q}$.

(b) We shall now show that there exists $z \in \mathbb{R} \setminus \mathbb{Q}$ such that $x < z < y$. By (a), there exist $r_1, r_2 \in \mathbb{Q}$ such that $x < r_1 < r_2 < y$. The number

$$z = r_1 + \frac{r_2 - r_1}{\sqrt{2}}$$

is clearly irrational and satisfies $r_1 < z < r_2$. ◯

1.4. Countability

In this account, we treat intuitively the distinction between finite and infinite sets. A set is finite if it contains a finite number of elements. To treat infinite sets, our starting point is the set $\mathbb{N}$ of all natural numbers, an example of an infinite set.

DEFINITION. A set X is said to be countably infinite if there exists a bijective mapping from X to $\mathbb{N}$. A set X is said to be countable if it is finite or countably infinite.

REMARK. Suppose that X is countably infinite. Then we can write

$$X = \{x_1, x_2, x_3, \ldots\}.$$

Here we understand that there is a bijective mapping $\phi : X \to \mathbb{N}$ where $\phi(x_n) = n$ for every $n \in \mathbb{N}$.

THEOREM 1E. *A countable union of countable sets is countable.*

PROOF. Let I be a countable index set, where for each $i \in I$, the set X_i is countable. Either (a) I is finite; or (b) I is countably infinite. We shall only consider (b), since (a) needs only minor modification. Since I is countably infinite, there exists a bijective mapping from I to $\mathbb{N}$. We may therefore assume, without loss of generality, that $I = \mathbb{N}$. For each $n \in \mathbb{N}$, since X_n is countable, we may write

$$X_n = \{a_{n1}, a_{n2}, a_{n3}, \ldots\},$$

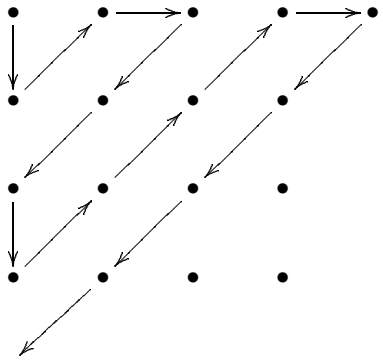
with the convention that if X_n is finite, then the sequence $a_{n1}, a_{n2}, a_{n3}, \ldots$ is constant from some point onwards. Hence we have a doubly infinite array

$$\begin{array}{cccc} a_{11} & a_{12} & a_{13} & \ldots \\ a_{21} & a_{22} & a_{23} & \ldots \\ a_{31} & a_{32} & a_{33} & \ldots \\ \vdots & \vdots & \vdots & \ddots \end{array}$$

of elements of the set

$$X = \bigcup_{n \in \mathbb{N}} X_n.$$

We now list these elements in the order indicated by



but discarding duplicates. If X is infinite, the above clearly gives rise to a bijection from X to $\mathbb{N}$. ◯

EXAMPLE 1.4.1. The set $\mathbb{Z}$ is countable; simply note that $\mathbb{Z} = \mathbb{N} \cup \{0\} \cup \{-1, -2, -3, \ldots\}$.

THEOREM 1F. *The set $\mathbb{Q}$ is countable.*

PROOF. Any $x \in \mathbb{Q}$ can be written in the form p/q, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. For every $n \in \mathbb{N}$, the set $Q_n = \{p/n : p \in \mathbb{Z}\}$ is countable (why?). Clearly

$$\mathbb{Q} = \bigcup_{n \in \mathbb{N}} Q_n.$$

The result follows from Theorem 1E. ◯

Suppose that two sets X_1 and X_2 are both countably infinite. Since both can be mapped to $\mathbb{N}$ bijectively, it follows that each can be mapped to the other bijectively. In this case, we say that the two sets X_1 and X_2 have the same cardinality. Cardinality can be considered as a way of measuring size. If there exists a one-to-one mapping from X_1 to X_2 and no one-to-one mapping from X_2 to X_1, then we say that X_2 has greater cardinality than X_1. For example, $\mathbb{N}$ and $\mathbb{Q}$ have the same cardinality. We shall now show that $\mathbb{R}$ has greater cardinality than $\mathbb{Q}$.

We shall first of all need an intermediate result.

THEOREM 1G. *Any subset of a countable set is countable.*

PROOF. Let X be a countable set. If X is finite, then the result is trivial. We therefore assume that X is countably infinite, so that we can write

$$X = \{x_1, x_2, x_3, \ldots\}.$$

Let Y be a subset of X. If Y is finite, then the result is trivial. If Y is countably infinite, then we can write

$$Y = \{x_{n_1}, x_{n_2}, x_{n_3}, \ldots\},$$

where

$$n_1 = \min\{n \in \mathbb{N} : x_n \in Y\},$$

and where, for every $p \geq 2$,

$$n_p = \min\{n > n_{p-1} : x_n \in Y\}.$$

The result follows. ◯

THEOREM 1H. *The set $\mathbb{R}$ is not countable.*

PROOF. In view of Theorem 1G, it suffices to show that $[0,1)$ is not countable. Suppose on the contrary that $[0,1)$ is countable. Then we can write

$$[0,1) = \{x_1, x_2, x_3, \ldots\}. \tag{1}$$

For each $n \in \mathbb{N}$, we express x_n in decimal notation in the form

$$x_n = .x_{n1}x_{n2}x_{n3}\ldots,$$

where for each $k \in \mathbb{N}$, the digit $x_{nk} \in \{0, 1, 2, \ldots, 9\}$. Note that this expression may not be unique, but it does not matter, as we simply choose one. We now have

$$\begin{aligned} x_1 &= .x_{11}x_{12}x_{13}\ldots, \\ x_2 &= .x_{21}x_{22}x_{23}\ldots, \\ x_3 &= .x_{31}x_{32}x_{33}\ldots, \\ &\vdots \end{aligned}$$

Let $y = .y_1y_2y_3\ldots$, where for each $n \in \mathbb{N}$, $y_n \in \{0, 1, 2, \ldots, 9\}$ and $y_n \equiv x_{nn} + 5 \pmod{10}$. Then clearly $y \neq x_n$ for any $n \in \mathbb{N}$. But $y_n \in [0,1)$, contradicting (1). ◯

Note that the set $\mathbb{R} \setminus \mathbb{Q}$ of all irrational numbers is not countable. It follows that in the sense of cardinality, there are far more irrational numbers than rational numbers.