INTRODUCTION TO LEBESGUE INTEGRATION

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Chapter 3

POINT SETS

3.1. Open and Closed Sets

To study a Riemann integral, one needs to subdivide the interval of integration into a finite number of subintervals. In Lebesgue's approach, the interval is subdivided into more general sets called measurable sets. In 1902, Lebesgue gave a definition of measure for point sets and used this to develop his integral.

Since then, measure theory and integration theory have both been generalized and modified. It is now possible to introduce the Lebesgue integral with very little reference to measure theory, but focusing directly on functions and their integrals instead.

We shall attempt here to give an account of this approach. The only concept from measure theory that we shall need is that of sets of measure zero. In this chapter, we shall cover some basic results on point sets for later use.

Definition. Suppose that $S \subseteq \mathbb{R}$ is given. A point $x \in S$ is said to be an interior point of S if there exists $\epsilon > 0$ such that the open interval $(x-\epsilon, x+\epsilon) \subseteq S$.

Definition. A set $G \subseteq \mathbb{R}$ is said to be open if every point of G is an interior point of G.

Remark. It is quite common to denote open sets by G after the German word "Gebiet".

Example 3.1.1. The interval $(0,1)$ is open. For any given $x \in (0,1)$, we can choose $\epsilon = \min\{x, 1-x\}$. Then $\epsilon \le x$ and $\epsilon \le 1-x$, so that $0 \le x-\epsilon < x+\epsilon \le 1$, whence $(x-\epsilon, x+\epsilon) \subseteq (0,1)$.

Example 3.1.2. The interval $[0,1]$ is not open, since clearly the point 0 is not an interior point of $[0,1]$.

EXAMPLE 3.1.3. The sets $\emptyset$ and $\mathbb{R}$ are both open.

We have the following two simple results.

THEOREM 3A. *The union of any collection of open sets in $\mathbb{R}$ is open.*

THEOREM 3B. *The intersection of any finite collection of open sets in $\mathbb{R}$ is open.*

REMARK. Note that Theorem 3B cannot be extended to infinite collections. Note, for example, that $G_n = (-1/n, 1/n)$ is open for every $n \in \mathbb{N}$. On the other hand,

$$\bigcap_{n=1}^{\infty} G_n = \{0\}$$

is not open. The reader is advised to study the proof of Theorem 3B below and try to pinpoint where the proof fails when the collection is infinite.

PROOF OF THEOREM 3A. Suppose that $\mathcal{G}$ is a collection of open sets in $\mathbb{R}$. Denote by U their union. Suppose that $x \in U$. Then $x \in G$ for some $G \in \mathcal{G}$. Since G is open, it follows that x is an interior point of G, and so there exists $\epsilon > 0$ such that

$$(x - \epsilon, x + \epsilon) \subseteq G \subseteq U.$$

It follows that x is an interior point of U. ◯

PROOF OF THEOREM 3B. Suppose that the open sets are $G_1, \ldots, G_n$. Denote by V their intersection. Suppose that $x \in V$. Then $x \in G_k$ for every $k = 1, \ldots, n$. Since G_k is open, it follows that x is an interior point of G_k, and so there exists $\epsilon_k > 0$ such that

$$(x - \epsilon_k, x + \epsilon_k) \subseteq G_k.$$

Now let $\epsilon = \min\{\epsilon_1, \ldots, \epsilon_n\} > 0$. Then for every $k = 1, \ldots, n$, we have

$$(x - \epsilon, x + \epsilon) \subseteq (x - \epsilon_k, x + \epsilon_k) \subseteq G_k,$$

so that

$$(x - \epsilon, x + \epsilon) \subseteq G_1 \cap \ldots \cap G_n = V.$$

It follows that x is an interior point of V. ◯

The following result gives a characterization of all open sets in $\mathbb{R}$.

THEOREM 3C. *Every open set $G \in \mathbb{R}$ is a countable union of pairwise disjoint open intervals in $\mathbb{R}$.*

PROOF. For every $x \in G$, let I_x denote the largest open interval in $\mathbb{R}$ satisfying $x \in I_x \subseteq G$. Suppose now that $x, y \in G$ and $I_x \cap I_y \neq \emptyset$. Then $I_x \cup I_y$ is also an open interval. Furthermore,

$$I_x \subseteq I_x \cup I_y \subseteq G \qquad \text{and} \qquad I_y \subseteq I_x \cup I_y \subseteq G.$$

From the definition of I_x and I_y, we must have $I_x = I_x \cup I_y$ and $I_y = I_x \cup I_y$, so that $I_x = I_y$. We therefore conclude that I_x and I_y are either disjoint or equal. It follows that G is a union of disjoint open intervals in $\mathbb{R}$. Write

$$G = \bigcup_{I \in \mathcal{C}} I.$$

It remains to show that the collection $\mathcal{C}$ is countable. Note that every interval $I \in \mathcal{C}$ contains a rational number x_I. We can now construct a bijective mapping $\phi : \mathcal{C} \to \{x_I : I \in \mathcal{C}\}$ by writing $\phi(I) = x_I$ for every $I \in \mathcal{C}$; in other words, we identify each interval I with a rational number it contains. Clearly $\{x_I : I \in \mathcal{C}\} \subseteq \mathbb{Q}$, and so must be countable. It follows that $\mathcal{C}$ is countable. ◯

Definition. Suppose that $S \subseteq \mathbb{R}$ is given. A point $x \in \mathbb{R}$ is said to be a limit point of S if it is the limit of a sequence in S.

Definition. A set $F \subseteq \mathbb{R}$ is said to be closed if it contains all its limit points.

Remark. It is quite common to denote closed sets by F after the French word “fermé”.

Example 3.1.4. The interval $(0, 1)$ is not closed. The sequence $1/n$ is in $(0, 1)$, but its limit 0 is not.

Example 3.1.5. The interval $[0, 1]$ is closed. If x_n is a convergent sequence in $[0, 1]$, then its limit x must satisfy $0 \le x \le 1$, so that $x \in [0, 1]$.

Example 3.1.6. The sets $\emptyset$ and $\mathbb{R}$ are both closed. These are examples of sets which are both open and closed.

We have the following useful result on open and closed sets.

THEOREM 3D. *A set $F \subseteq \mathbb{R}$ is closed if and only if its complement $F' = \mathbb{R} \setminus F$ is open.*

Proof. ($\Rightarrow$) Suppose that F is closed. For every $x \in F'$, x is not a limit point of F, so that no sequence in F converges to x. Hence there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \cap F = \emptyset$, so that $(x-\epsilon, x+\epsilon) \subseteq F'$.

($\Leftarrow$) Suppose that $x \notin F$. Then $x \in F'$. Since F' is open, it follows that there exists $\epsilon > 0$ such that $(x-\epsilon, x+\epsilon) \subseteq F'$, so that $(x-\epsilon, x+\epsilon) \cap F = \emptyset$. Hence no sequence in F converges to x, and so x is not a limit point of F. It now follows that F must contain all its limit points. ◯

Using Theorem 3D, the following two results follow immediately from Theorems 3A and 3B respectively.

THEOREM 3E. *The intersection of any collection of closed sets in $\mathbb{R}$ is closed.*

THEOREM 3F. *The union of any finite collection of closed sets in $\mathbb{R}$ is closed.*

Proof of Theorems 3E and 3F. Note simply De Morgan's law, that

$$\bigcap_{F \in \mathcal{F}} F = \mathbb{R} \setminus \left(\bigcup_{F \in \mathcal{F}} (\mathbb{R} \setminus F) \right)$$

for any collection $\mathcal{F}$ of sets in $\mathbb{R}$. ◯

Our aim is to establish the following important result.

THEOREM 3G. (CANTOR INTERSECTION THEOREM) *Suppose that the sequence of sets $F_n \subseteq \mathbb{R}$ satisfies the following conditions:*
(a) For every $n \in \mathbb{N}$, $F_n \neq \emptyset$.
(b) For every $n \in \mathbb{N}$, $F_{n+1} \subseteq F_n$.
(c) For every $n \in \mathbb{N}$, F_n is closed.
(d) F_1 is bounded.
Then the intersection

$$\bigcap_{n=1}^{\infty} F_n$$

is closed and non-empty.

To prove Theorem 3G, we need some results on real sequences.

DEFINITION. A sequence $x_n \in \mathbb{R}$ is said to be increasing if $x_{n+1} \geq x_n$ for every $n \in \mathbb{N}$. A sequence $x_n \in \mathbb{R}$ is said to be decreasing if $x_{n+1} \leq x_n$ for every $n \in \mathbb{N}$. A sequence $x_n \in \mathbb{R}$ is said to be monotonic if it is increasing or decreasing.

THEOREM 3H. Consider a sequence $x_n \in \mathbb{R}$.
(a) Suppose that x_n is increasing and bounded above. Then x_n is convergent.
(b) Suppose that x_n is decreasing and bounded below. Then x_n is convergent.

PROOF. We shall only prove (a), as the proof of (b) is similar. Since x_n is bounded above, let

$$M = \sup\{x_n : n \in \mathbb{N}\}.$$

We shall show that $x_n \to M$ as $n \to \infty$. Given any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $x_N > M - \epsilon$. Since x_n is increasing, it follows that for every $n > N$, we have

$$M - \epsilon < x_N \leq x_n \leq M < M + \epsilon,$$

so that $|x_n - M| < \epsilon$. The result follows. ◯

DEFINITION. Consider a sequence $x_n \in \mathbb{R}$. Suppose that $n_k \in \mathbb{N}$ for every $k \in \mathbb{N}$. Suppose further that

$$1 \leq n_1 < n_2 < n_3 < \ldots < n_k < \ldots.$$

Then the sequence x_{n_k} is called a subsequence of the sequence x_n.

EXAMPLE 3.1.7. The sequence of all even natural numbers is a subsequence of the sequence of all natural numbers. Here, note that $x_n = n$ for every $n \in \mathbb{N}$ and $n_k = 2k$ for every $k \in \mathbb{N}$.

EXAMPLE 3.1.8. The sequence $3, 5, 7, 11, \ldots$ of all odd primes is a subsequence of the sequence $1, 3, 5, 7, \ldots$ of all odd natural numbers. Here $x_n = 2n - 1$ for every $n \in \mathbb{N}$. Also $x_{n_1} = 3 = x_2$, $x_{n_2} = 5 = x_3$, $x_{n_3} = 7 = x_4$, and so on, so that $n_1 = 2$, $n_2 = 3$, $n_3 = 4$, and so on.

THEOREM 3J. Any sequence $x_n \in \mathbb{R}$ has a monotonic subsequence.

PROOF. We shall call $n \in \mathbb{N}$ a "peak" point if $x_m \leq x_n$ for every $m \geq n$. Then there are two cases:

(a) There are infinitely many peak points $n_1 < n_2 < \ldots < n_k < \ldots$. Then clearly

$$x_{n_1} \geq x_{n_2} \geq \ldots \geq x_{n_k} \geq \ldots,$$

and we have a decreasing subsequence.

(b) There are finitely many or no peak points. In this case, let $n_1 = N+1$ where N is the largest peak point, or $n_1 = 1$ if there are no peak points. Then n_1 is not a peak point, so there exists $n_2 > n_1$ such that $x_{n_2} > x_{n_1}$. Then n_2 is not a peak point, so there exists $n_3 > n_2$ such that $x_{n_3} > x_{n_2}$. Proceeding inductively, we conclude that there exists a sequence

$$n_1 < n_2 < \ldots < n_k < \ldots$$

of natural numbers such that

$$x_{n_1} < x_{n_2} < \ldots < x_{n_k} < \ldots,$$

and we have an increasing subsequence. ○

THEOREM 3K. (BOLZANO-WEIERSTRASS THEOREM) *Any bounded sequence $x_n \in \mathbb{R}$ has a convergent subsequence.*

PROOF. By Theorem 3J, the sequence x_n has a monotonic subsequence. Clearly this subsequence is bounded. The result now follows from Theorem 3H. ○

We can now prove the Cantor intersection theorem.

PROOF OF THEOREM 3G. The set

$$F = \bigcap_{n=1}^{\infty} F_n$$

is closed, in view of Theorem 3E. It remains to find a point $x \in F$. For every $n \in \mathbb{N}$, choose a point $x_n \in F_n$. The sequence x_n is clearly bounded, so it follows from the Bolzano-Weierstrass theorem that it has a convergent subsequence x_{n_k}. Suppose that $x_{n_k} \to x$ as $k \to \infty$. To show that $x \in F$, it suffices to show that $x \in F_n$ for every $n \in \mathbb{N}$. Note that in view of hypothesis (b), we have, for every $n \in \mathbb{N}$, that

$$x_n, x_{n+1}, x_{n+2}, \ldots \in F_n.$$

It follows that x is a limit point of F_n. Since F_n is closed, it follows that $x \in F_n$. This completes the proof of Theorem 3G. ○

3.2. Sets of Measure Zero

Our study of the Lebesgue integral will depend crucially on the notion of sets of measure zero in $\mathbb{R}$.

DEFINITION. A set $S \subseteq \mathbb{R}$ is said to have measure zero if, for every $\epsilon > 0$, there exists a countable collection $\mathcal{C}$ of open intervals I such that

$$S \subseteq \bigcup_{I \in \mathcal{C}} I \qquad \text{and} \qquad \sum_{I \in \mathcal{C}} \mu(I) < \epsilon,$$

where, for every $I \in \mathcal{C}$, $\mu(I)$ denotes the length of the interval I. In other words, the set S can be covered by a countable union of open intervals of arbitrarily small total length.

REMARK. The argument in the remainder of this section depends on the use of a convergent series of positive terms. For the sake of convenience, we have chosen the series

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

In fact, the argument will work with any convergent series of positive terms. We do not even need to know its sum, except for the fact that it is finite and positive.

EXAMPLE 3.2.1. We shall show that the set $\mathbb{Q}$ has measure zero. Note that $\mathbb{Q}$ is countable, so that we can write

$$\mathbb{Q} = \{x_1, x_2, x_3, \ldots\}.$$

Let $\epsilon > 0$ be given. For every $n \in \mathbb{N}$, let

$$I_n = \left(x_n - \frac{\epsilon}{2^{n+2}}, x_n + \frac{\epsilon}{2^{n+2}} \right).$$

Then clearly

$$\mathbb{Q} \subseteq \bigcup_{n=1}^{\infty} I_n \qquad \text{and} \qquad \sum_{n=1}^{\infty} \mu(I_n) = \frac{\epsilon}{2} \sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{\epsilon}{2} < \epsilon.$$

In fact, we have all but proved the following result.

THEOREM 3L. *Every countable set in $\mathbb{R}$ has measure zero.*

A similar idea enables us to prove the following result.

THEOREM 3M. *A countable union of sets of measure zero in $\mathbb{R}$ has measure zero.*

PROOF. We shall show that a countably infinite union of sets of measure zero in $\mathbb{R}$ has measure zero. The case of a finite union needs only minor modification. Suppose that for every $n \in \mathbb{N}$, the set $S_n \subseteq \mathbb{R}$ has measure zero. Given any $\epsilon > 0$, there exists a countable collection $\mathcal{C}_n$ of open intervals I such that

$$S_n \subseteq \bigcup_{I \in \mathcal{C}_n} I \qquad \text{and} \qquad \sum_{I \in \mathcal{C}_n} \mu(I) < \frac{\epsilon}{2^n}.$$

Let

$$\mathcal{C} = \bigcup_{n=1}^{\infty} \mathcal{C}_n.$$

Then $\mathcal{C}$ is countable by Theorem 1E. Clearly

$$\bigcup_{n=1}^{\infty} S_n \subseteq \bigcup_{n=1}^{\infty} \bigcup_{I \in \mathcal{C}_n} I = \bigcup_{I \in \mathcal{C}} I \qquad \text{and} \qquad \sum_{I \in \mathcal{C}} \mu(I) \le \sum_{n=1}^{\infty} \sum_{I \in \mathcal{C}_n} \mu(I) < \epsilon \sum_{n=1}^{\infty} \frac{1}{2^n} = \epsilon.$$

The result follows. ◯

DEFINITION. A property $P(x)$ is said to hold for almost all $x \in S$ if $P(x)$ fails to hold for at most a set of measure zero in S.

3.3. Compact Sets

DEFINITION. A set $S \subseteq \mathbb{R}$ is said to be compact if and only if, for every collection $\mathcal{C}$ of open intervals I such that

$$S \subseteq \bigcup_{I \in \mathcal{C}} I,$$

there exists a finite subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that

$$S \subseteq \bigcup_{I \in \mathcal{C}_0} I.$$

In other words, every open covering of S can be achieved by a finite subcovering.

Our main task in this section is to establish the following important result.

THEOREM 3N. (HEINE-BOREL THEOREM) *Suppose that $F \subseteq \mathbb{R}$ is bounded and closed. Then F is compact.*

PROOF. We need to show that for every collection $\mathcal{C}$ of open intervals I such that

$$F \subseteq \bigcup_{I \in \mathcal{C}} I,$$

there exists a finite subcollection $\mathcal{C}_0 \subseteq \mathcal{C}$ such that

$$F \subseteq \bigcup_{I \in \mathcal{C}_0} I.$$

We shall achieve this by first (a) reducing $\mathcal{C}$ to a countable subcollection $\mathcal{C}' \subseteq \mathcal{C}$; and then (b) reducing $\mathcal{C}'$ to a finite subcollection $\mathcal{C}_0 \subseteq \mathcal{C}'$.

(a) Let $\mathcal{Q}$ denote the collection of all open intervals in $\mathbb{R}$ with rational midpoints and lengths. Then $\mathcal{Q}$ is countable (why?), so that we can write

$$\mathcal{Q} = \{J_1, J_2, J_3, \ldots\}.$$

Suppose that $x \in F$. Then there exists $I \in \mathcal{C}$ such that $x \in I$. It is easy to see that we can find an interval $J_{n(x)} \in \mathcal{Q}$ such that

$$x \in J_{n(x)} \subseteq I. \tag{1}$$

Clearly

$$F \subseteq \bigcup_{\substack{n=1 \\ n=n(x) \text{ for some } x \in F}}^{\infty} J_n.$$

For every $n \in \mathbb{N}$ for which $n = n(x)$ for some $x \in F$, we now find an interval $I_n \in \mathcal{C}$ for which $J_n \subseteq I_n$; this is possible in view of (1). Then

$$F \subseteq \bigcup_{\substack{n=1 \\ n=n(x) \text{ for some } x \in F}}^{\infty} I_n.$$

(b) Suppose that

$$F \subseteq \bigcup_{I \in \mathcal{C}'} I.$$

The result is immediate if $\mathcal{C}'$ is finite, so we assume, without loss of generality, that $\mathcal{C}'$ is countably infinite. We can therefore write

$$\mathcal{C}' = \{I_1, I_2, I_3, \ldots\},$$

so that

$$F \subseteq \bigcup_{k=1}^{\infty} I_k.$$

For every $n \in \mathbb{N}$, the set

$$G_n = \bigcup_{k=1}^{n} I_k$$

is open, in view of Theorem 3A. We shall show that there exists $n \in \mathbb{N}$ such that $F \subseteq G_n$. For every $n \in \mathbb{N}$, consider the set

$$F_n = F \cap (\mathbb{R} \setminus G_n).$$

To complete the proof, it clearly suffices to show that $F_n = \emptyset$ for some $n \in \mathbb{N}$. Suppose, on the contrary, that $F_n \neq \emptyset$ for every $n \in \mathbb{N}$. Note that for every $n \in \mathbb{N}$, the set F_n is closed and bounded. Furthermore, $F_{n+1} \subseteq F_n$ for every $n \in \mathbb{N}$. It follows from the Cantor intersection theorem that

$$\bigcap_{n=1}^{\infty} F_n \neq \emptyset.$$

Hence there exists $x \in F$ such that $x \notin I_k$ for every $k \in \mathbb{N}$, clearly a contradiction. ◯

Remark. Part (a) of Theorem 3N is sometimes known as the Lindelöf covering theorem.