INTRODUCTION TO LEBESGUE INTEGRATION

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Chapter 4

THE LEBESGUE INTEGRAL

4.1. Step Functions on an Interval

The first step in our definition of the Lebesgue integral concerns step functions. In this section, we formulate a definition of the Lebesgue integral for step functions in terms of Riemann integrals, and study some of its properties.

Definition. Suppose that $A, B \in \mathbb{R}$ and $A < B$. A function $s : [A, B] \to \mathbb{R}$ is called a step function on $[A, B]$ if there exist a dissection $A = x_0 < x_1 < \ldots < x_n = B$ of $[A, B]$ and numbers $c_1, \ldots, c_n \in \mathbb{R}$ such that for every $k = 1, \ldots, n$, we have $s(x) = c_k$ for every $x \in (x_{k-1}, x_k)$.

Remark. Note that we have not imposed any conditions on $s(x_k)$ for any $k = 0, 1, \ldots, n$, except that they are real-valued. This is in view of the fact that a Riemann integral is unchanged if we alter the value of the function at a finite number of points.

For every $k = 1, \ldots, n$, the integral

$$\int_{x_{k-1}}^{x_k} s(x)\,\mathrm{d}x = c_k(x_k - x_{k-1})$$

in the sense of Riemann. Also the integral

$$\int_A^B s(x)\,\mathrm{d}x = \sum_{k=1}^{n} c_k(x_k - x_{k-1}) \tag{1}$$

in the sense of Riemann, and is in fact independent of the choice of the dissection of $[A, B]$, provided that $s(x)$ is constant in any open subinterval arising from the dissection.

We now make a simple generalization.

DEFINITION. Suppose that $I \subseteq \mathbb{R}$ is an interval. We say that a function $s : I \to \mathbb{R}$ is a step function on I, denoted by $s \in \mathcal{S}(I)$, if there exists a finite subinterval $(A, B) \subseteq I$ such that $s : [A, B] \to \mathbb{R}$ is a step function on $[A, B]$ and $s(x) = 0$ for every $x \in I \setminus [A, B]$. Furthermore, the integral

$$\int_I s(x)\,\mathrm{d}x \tag{2}$$

is defined by the integral of s over $[A, B]$ given by (1).

REMARKS. (1) Note that in the above definition, the function $s : I \to \mathbb{R}$ may not be defined at $x = A$ and/or $x = B$. In this case, we may assign $s(A)$ and $s(B)$ arbitrary finite values, and note that (1) is not affected by this process.

(2) Of course, the choice of the interval $[A, B]$ may not be unique. However, in view of the requirement that $s(x) = 0$ for every $x \in I \setminus [A, B]$, it is not difficult to see that the value of the integral (2) is independent of the choice of such $[A, B]$.

The following theorem can be deduced directly from the definitions.

THEOREM 4A. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $s, t \in \mathcal{S}(I)$. Then*

(a) $s + t \in \mathcal{S}(I)$ and $\displaystyle\int_I (s(x) + t(x))\,\mathrm{d}x = \int_I s(x)\,\mathrm{d}x + \int_I t(x)\,\mathrm{d}x$;

(b) for every $c \in \mathbb{R}$, $cs \in \mathcal{S}(I)$ and $\displaystyle\int_I cs(x)\,\mathrm{d}x = c\int_I s(x)\,\mathrm{d}x$; and

(c) if $s(x) \leq t(x)$ for every $x \in I$, then $\displaystyle\int_I s(x)\,\mathrm{d}x \leq \int_I t(x)\,\mathrm{d}x$.

PROOF. (a) From the definition, there exist intervals $(A_1, B_1) \subseteq I$ and $(A_2, B_2) \subseteq I$ such that s and t are step functions on $[A_1, B_1]$ and $[A_2, B_2]$ respectively,

$$\int_I s(x)\,\mathrm{d}x = \int_{A_1}^{B_1} s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I t(x)\,\mathrm{d}x = \int_{A_2}^{B_2} t(x)\,\mathrm{d}x,$$

and that $s(x) = 0$ for every $x \in I \setminus [A_1, B_1]$ and $t(x) = 0$ for every $x \in I \setminus [A_2, B_2]$. Furthermore, the integrals

$$\int_{A_1}^{B_1} s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_{A_2}^{B_2} t(x)\,\mathrm{d}x$$

are in the sense of Riemann. Now let $A = \min\{A_1, A_2\}$ and $B = \max\{B_1, B_2\}$. Then

$$(A_1, B_1) \subseteq (A, B) \subseteq I \qquad \text{and} \qquad (A_2, B_2) \subseteq (A, B) \subseteq I.$$

Furthermore, it is easy to see that both s and t are step functions in $[A, B]$, and that $s(x) = t(x) = 0$ for every $x \in I \setminus [A, B]$. Hence

$$\int_I s(x)\,\mathrm{d}x = \int_A^B s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I t(x)\,\mathrm{d}x = \int_A^B t(x)\,\mathrm{d}x. \tag{3}$$

Note also that the integrals

$$\int_A^B s(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_A^B t(x)\,\mathrm{d}x$$

are in the sense of Riemann. On the other hand, it is easily checked that $s+t$ is a step function on $[A,B]$, and that $s(x)+t(x)=0$ for every $x\in I\setminus[A,B]$. By definition, we have

$$\int_I (s(x)+t(x))\,\mathrm{d}x = \int_A^B (s(x)+t(x))\,\mathrm{d}x. \tag{4}$$

Note, however, that

$$\int_A^B (s(x)+t(x))\,\mathrm{d}x = \int_A^B s(x)\,\mathrm{d}x + \int_A^B t(x)\,\mathrm{d}x, \tag{5}$$

where the integrals in (5) are in the sense of Riemann. The result now follows on combining (3)–(5).

(b) From the definition, there exists an interval $(A,B)\subseteq I$ such that s is a step function on $[A,B]$,

$$\int_I s(x)\,\mathrm{d}x = \int_A^B s(x)\,\mathrm{d}x, \tag{6}$$

and that $s(x)=0$ for every $x\in I\setminus[A,B]$. Furthermore, the integral

$$\int_A^B s(x)\,\mathrm{d}x$$

is in the sense of Riemann. It is easy to see that cs is a step function on $[A,B]$, and that $cs(x)=0$ for every $x\in I\setminus[A,B]$. By definition, we have

$$\int_I cs(x)\,\mathrm{d}x = \int_A^B cs(x)\,\mathrm{d}x. \tag{7}$$

Note, however, that

$$\int_A^B cs(x)\,\mathrm{d}x = c\int_A^B s(x)\,\mathrm{d}x, \tag{8}$$

where the integrals in (8) are in the sense of Riemann. The result now follows on combining (6)–(8).

(c) We follow the argument in part (a) and note, instead, that

$$\int_A^B s(x)\,\mathrm{d}x \le \int_A^B t(x)\,\mathrm{d}x, \tag{9}$$

where the integrals in (9) are in the sense of Riemann. The result now follows on combining (3) and (9). ◯

THEOREM 4B. *Suppose that the interval $I\subseteq\mathbb{R}$ can be written in the form $I=I_1\cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $s\in\mathcal{S}(I)$. Then*

$$\int_I s(x)\,\mathrm{d}x = \int_{I_1} s(x)\,\mathrm{d}x + \int_{I_2} s(x)\,\mathrm{d}x.$$

PROOF. For $j=1,2$, let $\chi_j : I\to\mathbb{R}$ denote the characteristic function of the interval I_j. Then $s(x)=s(x)\chi_1(x)+s(x)\chi_2(x)$ for every $x\in I$, apart from possibly a finite number of exceptions (which do not affect the values of the integrals). Note now that $s(x)\chi_j(x)$ is a step function on I_1, I_2 and I, and that $s(x)\chi_j(x) = 0$ for every $x \in I \setminus I_j$. Furthermore, $s(x)\chi_j(x) = s(x)$ for every $x \in I_j$. It follows that

$$\int_I s(x)\,\mathrm{d}x = \int_I (s(x)\chi_1(x)+s(x)\chi_2(x))\,\mathrm{d}x = \int_I s(x)\chi_1(x)\,\mathrm{d}x+\int_I s(x)\chi_2(x)\,\mathrm{d}x = \int_{I_1} s(x)\,\mathrm{d}x+\int_{I_2} s(x)\,\mathrm{d}x$$

as required. ○

4.2. Upper Functions on an Interval

The second step in our definition of the Lebesgue integral concerns extending the definition of the Lebesgue integral for step functions to a larger collection which we shall call the upper functions. In this section, we formulate a definition of the Lebesgue integral for upper functions by studying sequences of step functions, and study some of its properties.

Definition. Suppose that $S \subseteq \mathbb{R}$. A sequence of functions $f_n : S \to \mathbb{R}$ is said to be increasing on S if $f_{n+1}(x) \ge f_n(x)$ for every $n \in \mathbb{N}$ and every $x \in S$. A sequence of functions $f_n : S \to \mathbb{R}$ is said to be decreasing on S if $f_{n+1}(x) \le f_n(x)$ for every $n \in \mathbb{N}$ and every $x \in S$.

Definition. Suppose that $u : I \to \mathbb{R}$ is a function defined on an interval $I \subseteq \mathbb{R}$. Suppose further that there exists a sequence of step functions $s_n \in \mathcal{S}(I)$ satisfying the following conditions:
(a) The sequence $s_n : I \to \mathbb{R}$ is increasing on I.
(b) $s_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$.
(c) $\displaystyle\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x$ exists.

Then we say that the sequence of step functions $s_n \in \mathcal{S}(I)$ generates u, and that u is an upper function on I, denoted by $u \in \mathcal{U}(I)$. Furthermore, we define the integral of u over I by

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x. \tag{10}$$

The validity of the definition is justified by the following result.

THEOREM 4C. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $u \in \mathcal{U}(I)$. Suppose further that both sequences $s_n \in \mathcal{S}(I)$ and $t_n \in \mathcal{S}(I)$ generate u. Then*

$$\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x.$$

Theorem 4C is a simple consequence of the following result on step functions.

THEOREM 4D. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence $t_n \in \mathcal{S}(I)$ satisfies the following conditions:*
(a) The sequence $t_n : I \to \mathbb{R}$ is increasing on I.
(b) There exists a function $u : I \to \mathbb{R}$ such that $t_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$.
(c) $\displaystyle\lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x$ exists.

Then for any $t \in \mathcal{S}(I)$ satisfying $t(x) \le u(x)$ for almost all $x \in I$, we have

$$\int_I t(x)\,\mathrm{d}x \le \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x.$$

Proof of Theorem 4C. Note that the sequence of step functions $t_n : I \to \mathbb{R}$ satisfies hypotheses (a) and (c) of Theorem 4D. Furthermore, since this sequence generates u, it follows that hypothesis (b) of Theorem 4D is satisfied. On the other hand, for every $m \in \mathbb{N}$, it is easy to see that $s_n(x) \leq u(x)$ for almost all $x \in I$. It now follows from Theorem 4D that for every $m \in \mathbb{N}$, we have

$$\int_I s_m(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x,$$

and so on letting $m \to \infty$, we have

$$\lim_{m\to\infty} \int_I s_m(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x$$

(note here that m and n are "dummy" variables). Reversing the roles of the two sequences, the opposite inequality

$$\lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x \leq \lim_{m\to\infty} \int_I s_m(x)\,\mathrm{d}x$$

can be established by a similar argument. The result follows immediately. ○

The main part of the proof of Theorem 4D can be summarized by the following result.

THEOREM 4E. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence $s_n \in \mathcal{S}(I)$ satisfies the following conditions:*
(a) The sequence $s_n : I \to \mathbb{R}$ is decreasing on I.
(b) $s_n(x) \geq 0$ for every $n \in \mathbb{N}$ and every $x \in I$.
(c) $s_n(x) \to 0$ as $n \to \infty$ for almost all $x \in I$.
Then

$$\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x = 0.$$

Proof. Since $s_1 \in \mathcal{S}(I)$, there exists $(A,B) \subseteq I$ such that $s_1(x) = 0$ for every $x \in I \setminus [A,B]$. For every $n \in \mathbb{N}$ and every $x \in I$, we clearly have $0 \leq s_n(x) \leq s_1(x)$, and so $s_n(x) = 0$ for every $x \in I \setminus [A,B]$. Since $s_n \in \mathcal{S}(I)$, it is a step function on $[A,B]$, and

$$\int_I s_n(x)\,\mathrm{d}x = \int_A^B s_n(x)\,\mathrm{d}x, \tag{11}$$

where the integral on the right hand side is in the sense of Riemann. Furthermore, there exists a dissection Δ_n of $[A,B]$ such that $s_n(x)$ is constant in any open subinterval arising from Δ_n. Let

$$\mathcal{D} = \bigcup_{n=1}^{\infty} \Delta_n$$

represent the collection of all dissection points. Since Δ_n is a finite set for every $n \in \mathbb{N}$, it follows that $\mathcal{D}$ is countable, and so has measure 0. Next, let

$$\mathcal{E} = \{x \in I : s_n(x) \not\to 0 \text{ as } n \to \infty\}$$

denote the set of exceptional points of non-convergence. By (c), $\mathcal{E}$ also has measure 0, so that the set

$$\mathcal{F} = \mathcal{D} \cup \mathcal{E}$$

has measure 0. Let $\epsilon > 0$ be given and fixed. Then there exists a countable collection of open intervals $\mathcal{F}_k$, where $k \in \mathcal{K}$, of total length less than ϵ, such that

$$\mathcal{F} \subseteq \bigcup_{k \in \mathcal{K}} \mathcal{F}_k.$$

Suppose now that $y \in [A, B] \setminus \mathcal{F}$. On the one hand, since $y \notin \mathcal{E}$, it follows that $s_n(y) \to 0$ as $n \to \infty$, so that there exists $N = N(y)$ such that $s_N(y) < \epsilon$. On the other hand, since $y \notin \mathcal{D}$, it follows that there is an open interval $\mathcal{I}(y)$ such that $y \in \mathcal{I}(y)$ and $s_N(x)$ is constant in $\mathcal{I}(y)$, so that $s_N(x) < \epsilon$ for every $x \in \mathcal{I}(y)$. Clearly the open intervals $\mathcal{I}(y)$, as y runs over $[A, B] \setminus \mathcal{F}$, together with the open intervals $\mathcal{F}_k$, where $k \in \mathcal{K}$, form an open covering of $[A, B]$. Since $[A, B]$ is compact, there is a finite subcovering

$$[A, B] \subseteq \left(\bigcup_{i=1}^{p} \mathcal{I}(y_i) \right) \cup \left(\bigcup_{j=1}^{q} \mathcal{F}_j \right).$$

Let $N_0 = \max\{N(y_1), \ldots, N(y_p)\}$. In view of (a), we clearly have

$$s_n(x) < \epsilon \qquad \text{for every } n > N_0 \text{ and } x \in \bigcup_{i=1}^{p} \mathcal{I}(y_i). \tag{12}$$

Write

$$\mathcal{T}_1 = \bigcup_{j=1}^{q} \mathcal{F}_j \qquad \text{and} \qquad \mathcal{T}_2 = [A, B] \setminus \mathcal{T}_1,$$

and note that both can be written as finite unions of disjoint intervals. For every $n \in \mathbb{N}$, since $s_n(x) = 0$ outside $[A, B]$, it follows that

$$\int_A^B s_n(x) \, \mathrm{d}x = \int_{\mathcal{T}_1} s_n(x) \, \mathrm{d}x + \int_{\mathcal{T}_2} s_n(x) \, \mathrm{d}x, \tag{13}$$

where all the integrals are in the sense of Riemann. We now estimate each of the integrals on the right hand side of (13). To estimate the integral over $\mathcal{T}_1$, let M denote an upper bound of $s_1(x)$ on $[A, B]$. Then $s_n(x) \le M$ for every $x \in \mathcal{T}_1$ (why?). On the other hand, note that the intervals $\mathcal{F}_k$ have total length less than ϵ. Hence

$$\int_{\mathcal{T}_1} s_n(x) \, \mathrm{d}x \le M\epsilon. \tag{14}$$

To estimate the integral over $\mathcal{T}_2$, note that

$$\mathcal{T}_2 \subseteq \bigcup_{i=1}^{p} \mathcal{I}(y_i).$$

It follows from (12) that $s_n(x) < \epsilon$ for every $n > N_0$ and $x \in \mathcal{T}_2$. On the other hand, note that $\mathcal{T}_2 \subseteq [A, B]$. Hence for every $n > N_0$,

$$\int_{\mathcal{T}_2} s_n(x) \, \mathrm{d}x \le \epsilon(B - A). \tag{15}$$

Combining (11) and (13)–(15), we conclude that for every $n > N_0$,

$$\int_I s_n(x) \, \mathrm{d}x \le (M + B - A)\epsilon.$$

The result follows. ◯

Proof of Theorem 4D. For every $n \in \mathbb{N}$ and every $x \in I$, write $s_n(x) = \max\{t(x) - t_n(x), 0\}$. Clearly $s_n(x) \geq 0$ for every $x \in I$. Since $t, t_n \in \mathcal{S}(I)$, it follows that $s_n \in \mathcal{S}(I)$. Since the sequence t_n is increasing on I, it follows that the sequence s_n is decreasing on I. Finally, since $t_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$, it follows that $s_n(x) \to \max\{t(x) - u(x), 0\}$ for almost all $x \in I$. It now follows from Theorem 4E that

$$\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x = 0. \tag{16}$$

On the other hand, clearly $s_n(x) \geq t(x) - t_n(x)$ for every $n \in \mathbb{N}$ and $x \in I$. It follows from Theorem 4A that

$$\int_I s_n(x)\,\mathrm{d}x \geq \int_I t(x)\,\mathrm{d}x - \int_I t_n(x)\,\mathrm{d}x. \tag{17}$$

The result now follows on letting $n \to \infty$ in (17) and combining with (16). ○

Corresponding to Theorem 4A, we have the following result.

THEOREM 4F. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $u, v \in \mathcal{U}(I)$. Then*

(a) $u + v \in \mathcal{U}(I)$ and $\displaystyle\int_I (u(x) + v(x))\,\mathrm{d}x = \int_I u(x)\,\mathrm{d}x + \int_I v(x)\,\mathrm{d}x$;

(b) for every non-negative $c \in \mathbb{R}$, $cu \in \mathcal{U}(I)$ and $\displaystyle\int_I cu(x)\,\mathrm{d}x = c\int_I u(x)\,\mathrm{d}x$;

(c) if $u(x) \leq v(x)$ for almost all $x \in I$, then $\displaystyle\int_I u(x)\,\mathrm{d}x \leq \int_I v(x)\,\mathrm{d}x$; and

(d) if $u(x) = v(x)$ for almost all $x \in I$, then $\displaystyle\int_I u(x)\,\mathrm{d}x = \int_I v(x)\,\mathrm{d}x$.

Proof. Since $u, v \in \mathcal{U}(I)$, there exist increasing sequences $s_n \in \mathcal{S}(I)$ and $t_n \in \mathcal{S}(I)$ of step functions such that $s_n(x) \to u(x)$ and $t_n(x) \to v(x)$ as $n \to \infty$ for almost all $x \in I$, and that

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I v(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x. \tag{18}$$

It follows that $s_n + t_n$ and cs_n for any $c \geq 0$ are increasing sequences of step functions on I. Furthermore, $s_n(x) + t_n(x) \to u(x) + v(x)$ and $cs_n(x) \to cu(x)$ as $n \to \infty$ for almost all $x \in I$. By definition, we have

$$\int_I (u(x) + v(x))\,\mathrm{d}x = \lim_{n\to\infty} \int_I (s_n(x) + t_n(x))\,\mathrm{d}x \qquad \text{and} \qquad \int_I cu(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I cs_n(x)\,\mathrm{d}x, \tag{19}$$

provided that the limits exist. In view of Theorem 4A, we have, for every $n \in \mathbb{N}$, that

$$\int_I (s_n(x) + t_n(x))\,\mathrm{d}x = \int_I s_n(x)\,\mathrm{d}x + \int_I t_n(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I cs_n(x)\,\mathrm{d}x = c\int_I s_n(x)\,\mathrm{d}x. \tag{20}$$

(a) and (b) now follow on letting $n \to \infty$ in (20) and combining with (18) and (19). To prove (c), note that for every $m \in \mathbb{N}$, we have

$$s_m(x) \leq u(x) \leq v(x) = \lim_{n\to\infty} t_n(x)$$

for almost all $x \in I$. It follows from Theorem 4D that

$$\int_I s_m(x)\,\mathrm{d}x \leq \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x = \int_I v(x)\,\mathrm{d}x.$$

(c) now follows on letting $m \to \infty$. To prove (d), note that we clearly have $u(x) \le v(x)$ and $v(x) \le u(x)$ for almost all $x \in I$. It follows from (c) that

$$\int_I u(x)\,\mathrm{d}x \le \int_I v(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I v(x)\,\mathrm{d}x \le \int_I u(x)\,\mathrm{d}x.$$

Equality therefore must hold. ◯

DEFINITION. Suppose that $S \subseteq \mathbb{R}$. For functions $f : S \to \mathbb{R}$ and $g : S \to \mathbb{R}$, we define the maximum and minimum functions $\max\{f,g\} : S \to \mathbb{R}$ and $\min\{f,g\} : S \to \mathbb{R}$ by writing

$$\max\{f,g\}(x) = \max\{f(x),g(x)\} \qquad \text{and} \qquad \min\{f,g\}(x) = \min\{f(x),g(x)\}$$

for every $x \in S$.

THEOREM 4G. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $u, v \in \mathcal{U}(I)$. Then so are $\max\{u,v\}$ and $\min\{u,v\}$.*

PROOF. Since $u, v \in \mathcal{U}(I)$, there exist increasing sequences $s_n \in \mathcal{S}(I)$ and $t_n \in \mathcal{S}(I)$ of step functions such that $s_n(x) \to u(x)$ and $t_n(x) \to v(x)$ as $n \to \infty$ for almost all $x \in I$. It is easy to see that $a_n = \max\{s_n, t_n\}$ and $b_n = \min\{s_n, t_n\}$ are increasing sequences of step functions on I, and that $a_n(x) \to \max\{u,v\}(x)$ and $b_n(x) \to \min\{u,v\}(x)$ as $n \to \infty$ for almost all $x \in I$. It remains to show that both sequences

$$\int_I a_n(x)\,\mathrm{d}x \qquad \text{and} \qquad \int_I b_n(x)\,\mathrm{d}x$$

are convergent. To establish the convergence of the sequence

$$\int_I b_n(x)\,\mathrm{d}x, \tag{21}$$

note that it is increasing. On the other hand, for every $n \in \mathbb{N}$, we have $b_n(x) \le s_n(x) \le u(x)$ for almost all $x \in I$. It follows from Theorem 4F(c) that

$$\int_I b_n(x)\,\mathrm{d}x \le \int_I u(x)\,\mathrm{d}x,$$

so that (21) is bounded above. Finally, it is not difficult to check that for every $n \in \mathbb{N}$, we have $a_n + b_n = s_n + t_n$, so that $a_n = s_n + t_n - b_n$. It follows from Theorem 4A that

$$\int_I a_n(x)\,\mathrm{d}x = \int_I s_n(x)\,\mathrm{d}x + \int_I t_n(x)\,\mathrm{d}x - \int_I b_n(x)\,\mathrm{d}x. \tag{22}$$

The convergence of the left hand side of (22) follows immediately from the convergence of the right hand side. ◯

Corresponding to Theorem 4B, we have the following result.

THEOREM 4H. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $u \in \mathcal{U}(I)$, and that $u(x) \ge 0$ for almost all $x \in I$. Then $u \in \mathcal{U}(I_1)$ and $u \in \mathcal{U}(I_2)$, and*

$$\int_I u(x)\,\mathrm{d}x = \int_{I_1} u(x)\,\mathrm{d}x + \int_{I_2} u(x)\,\mathrm{d}x.$$

This is complemented by the following result.

THEOREM 4J. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $u_1 \in \mathcal{U}(I_1)$ and $u_2 \in \mathcal{U}(I_2)$. Define the function $u : I \to \mathbb{R}$ by*

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in I_1, \\ u_2(x) & \text{if } x \in I \setminus I_1. \end{cases}$$

Then $u \in \mathcal{U}(I)$, and

$$\int_I u(x)\,\mathrm{d}x = \int_{I_1} u_1(x)\,\mathrm{d}x + \int_{I_2} u_2(x)\,\mathrm{d}x.$$

PROOF OF THEOREM 4H. Since $u \in \mathcal{U}(I)$, there exists an increasing sequence $s_n \in \mathcal{S}(I)$ of step functions such that $s_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$. Since $u(x) \geq 0$ for almost all $x \in I$, it is easy to see that $s_n^+ = \max\{s_n, 0\}$ is an increasing sequence of step functions on I, and that $s_n^+(x) \to u(x)$ for almost all $x \in I$. It follows that for every subinterval $J \subseteq I$, s_n^+ is an increasing sequence of step functions on J, and $s_n^+(x) \to u(x)$ for almost all $x \in J$. To show that $u \in \mathcal{U}(J)$, it remains to show that the sequence

$$\int_J s_n^+(x)\,\mathrm{d}x \tag{23}$$

is convergent. This follows easily on noting that the sequence (23) is increasing, and that

$$\int_J s_n^+(x)\,\mathrm{d}x \leq \int_I s_n^+(x)\,\mathrm{d}x \leq \int_I u(x)\,\mathrm{d}x,$$

so that it is bounded above. This proves that $u \in \mathcal{U}(I_1)$ and $u \in \mathcal{U}(I_2)$. To complete the proof, note that for every $n \in \mathbb{N}$, we have

$$\int_I s_n^+(x)\,\mathrm{d}x = \int_{I_1} s_n^+(x)\,\mathrm{d}x + \int_{I_2} s_n^+(x)\,\mathrm{d}x,$$

in view of Theorem 4B. The result now follows on letting $n \to \infty$. ◯

PROOF OF THEOREM 4J. Since $u_1 \in \mathcal{U}(I_1)$, there exists an increasing sequence s_n of step functions on I_1 such that $s_n(x) \to u_1(x)$ as $n \to \infty$ for almost all $x \in I_1$. Since $u_2 \in \mathcal{U}(I_2)$, there exists an increasing sequence t_n of step functions on I_2 such that $t_n(x) \to u_2(x)$ as $n \to \infty$ for almost all $x \in I_2$. For every $n \in \mathbb{N}$, define the function $a_n : I \to \mathbb{R}$ by writing

$$a_n(x) = \begin{cases} s_n(x) & \text{if } x \in I_1, \\ t_n(x) & \text{if } x \in I \setminus I_1. \end{cases}$$

It is easy to see that a_n is an increasing sequence of step functions on I, and that $a_n(x) \to u(x)$ as $n \to \infty$ for almost all $x \in I$. This proves that $u \in \mathcal{U}(I)$. To complete the proof, note that for every $n \in \mathbb{N}$, we have

$$\int_I a_n(x)\,\mathrm{d}x = \int_{I_1} s_n(x)\,\mathrm{d}x + \int_{I_2} t_n(x)\,\mathrm{d}x,$$

noting that $a_n(x) = t_n(x)$ for almost all $x \in I_2$. The result now follows on letting $n \to \infty$. ◯

4.3. Lebesgue Integrable Functions on an Interval

The final step in our definition of the Lebesgue integral concerns extending the definition of the Lebesgue integral for upper functions to a larger collection which we shall call the Lebesgue integrable functions.

Definition. Suppose that $f : I \to \mathbb{R}$ is a function defined on an interval $I \subseteq \mathbb{R}$. Suppose further that there exist upper functions $u : I \to \mathbb{R}$ and $v : I \to \mathbb{R}$ on I such that $f(x) = u(x) - v(x)$ for all $x \in I$. Then we say that f is a Lebesgue integrable function on I, denoted by $f \in \mathcal{L}(I)$. We also say that f is Lebesgue integrable over I, and define the integral of f over I by

$$\int_I f(x)\,\mathrm{d}x = \int_I u(x)\,\mathrm{d}x - \int_I v(x)\,\mathrm{d}x.$$

The validity of the definition is justified by the following simple result. The proof is left as an exercise.

THEOREM 4K. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that $u_1, v_1, u_2, v_2 \in \mathcal{U}(I)$, and that $u_1(x) - v_1(x) = u_2(x) - v_2(x)$ for every $x \in I$. Then*

$$\int_I u_1(x)\,\mathrm{d}x - \int_I v_1(x)\,\mathrm{d}x = \int_I u_2(x)\,\mathrm{d}x - \int_I v_2(x)\,\mathrm{d}x.$$

Corresponding to Theorems 4A and 4F, we have the following result. The proof is left as an exercise.

THEOREM 4L. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f, g \in \mathcal{L}(I)$. Then*

(a) $f + g \in \mathcal{L}(I)$ and $\displaystyle\int_I (f(x) + g(x))\,\mathrm{d}x = \int_I f(x)\,\mathrm{d}x + \int_I g(x)\,\mathrm{d}x$;

(b) for every $c \in \mathbb{R}$, $cf \in \mathcal{L}(I)$ and $\displaystyle\int_I cf(x)\,\mathrm{d}x = c\int_I f(x)\,\mathrm{d}x$;

(c) if $f(x) \geq 0$ for almost all $x \in I$, then $\displaystyle\int_I f(x)\,\mathrm{d}x \geq 0$;

(d) if $f(x) \geq g(x)$ for almost all $x \in I$, then $\displaystyle\int_I f(x)\,\mathrm{d}x \geq \int_I g(x)\,\mathrm{d}x$; and

(e) if $f(x) = g(x)$ for almost all $x \in I$, then $\displaystyle\int_I f(x)\,\mathrm{d}x = \int_I g(x)\,\mathrm{d}x$.

We now investigate some further properties of the Lebesgue integral.

THEOREM 4M. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Then so are $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ and $|f|$. Furthermore,*

$$\left|\int_I f(x)\,\mathrm{d}x\right| \leq \int_I |f(x)|\,\mathrm{d}x. \tag{24}$$

Proof. There exist $u, v \in \mathcal{U}(I)$ such that $f(x) = u(x) - v(x)$ for all $x \in I$. Then

$$f^+ = \max\{u - v, 0\} = \max\{u, v\} - v.$$

By Theorem 4G, $\max\{u, v\} \in \mathcal{U}(I)$. It follows that $f^+ \in \mathcal{L}(I)$. By Theorem 4L(a)(b), we also have $f^- = f^+ - f \in \mathcal{L}(I)$ and $|f| = f^+ + f^- \in \mathcal{L}(I)$. On the other hand, we have $-|f(x)| \leq f(x) \leq |f(x)|$ for every $x \in I$. The inequality (24) now follows from Theorem 4L(d). ○

THEOREM 4N. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f, g \in \mathcal{L}(I)$. Then so are $\max\{f, g\}$, $\min\{f, g\}$.*

Proof. Note that

$$\max\{f, g\} = \frac{f + g + |f - g|}{2} \qquad \text{and} \qquad \min\{f, g\} = \frac{f + g - |f - g|}{2}.$$

The result now follows from Theorem 4L(a)(b). ○

Corresponding to Theorems 4H and 4J, we have the following two results. The proofs are left as exercises.

THEOREM 4P. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $f \in \mathcal{L}(I)$. Then $f \in \mathcal{L}(I_1)$ and $f \in \mathcal{L}(I_2)$, and*

$$\int_I f(x)\,\mathrm{d}x = \int_{I_1} f(x)\,\mathrm{d}x + \int_{I_2} f(x)\,\mathrm{d}x.$$

THEOREM 4Q. *Suppose that the interval $I \subseteq \mathbb{R}$ can be written in the form $I = I_1 \cup I_2$, where the intervals I_1 and I_2 have no interior points in common. Suppose further that $f_1 \in \mathcal{L}(I_1)$ and $f_2 \in \mathcal{L}(I_2)$. Define the function $f : I \to \mathbb{R}$ by*

$$f(x) = \begin{cases} f_1(x) & \text{if } x \in I_1, \\ f_2(x) & \text{if } x \in I \setminus I_1. \end{cases}$$

Then $f \in \mathcal{L}(I)$, and

$$\int_I f(x)\,\mathrm{d}x = \int_{I_1} f_1(x)\,\mathrm{d}x + \int_{I_2} f_2(x)\,\mathrm{d}x.$$

We conclude this section by proving the following two results which are qualitative statements concerning the approximation of a Lebesgue integrable function by an upper function and by a step function respectively.

THEOREM 4R. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Then for every $\epsilon > 0$, there exist $u, v \in \mathcal{U}(I)$ satisfying the following conditions:*
(a) $f(x) = u(x) - v(x)$ for every $x \in I$;
(b) $v(x) \ge 0$ for almost all $x \in I$; and
(c) $\displaystyle\int_I v(x)\,\mathrm{d}x < \epsilon$.

Proof. There exist $u_1, v_1 \in \mathcal{U}(I)$ such that $f = u_1 - v_1$ on I. Suppose that v_1 is generated by the sequence of step functions $t_n \in \mathcal{S}(I)$. Since

$$\int_I v_1(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I t_n(x)\,\mathrm{d}x,$$

it follows that there exists $N \in \mathbb{N}$ such that

$$0 \le \int_I (v_1(x) - t_N(x))\,\mathrm{d}x = \left| \int_I v_1(x)\,\mathrm{d}x - \int_I t_N(x)\,\mathrm{d}x \right| < \epsilon.$$

Let $u = u_1 - t_N$ and $v = v_1 - t_N$ on I. Then it is easy to see that $u, v \in \mathcal{U}(I)$. Also (a) and (c) follow immediately. To show (b), note that the sequence t_n is increasing, and that $t_n(x) \to v_1(x)$ as $n \to \infty$ for almost all $x \in I$. ○

THEOREM 4S. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Then for every $\epsilon > 0$, there exist $s \in \mathcal{S}(I)$ and $g \in \mathcal{L}(I)$ satisfying the following conditions:*
(a) $f(x) = s(x) + g(x)$ for every $x \in I$;and
(b) $\displaystyle\int_I |g(x)|\,\mathrm{d}x < \epsilon$.

PROOF. By Theorem 4R, there exist $u, v \in \mathcal{U}(I)$ such that $f = u - v$ on I, $v(x) \geq 0$ for almost all $x \in I$, and

$$0 \leq \int_I v(x)\,\mathrm{d}x < \frac{\epsilon}{2}. \tag{25}$$

Suppose that u is generated by the sequence of step functions $s_n \in \mathcal{S}(I)$. Since

$$\int_I u(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x,$$

it follows that there exists $N \in \mathbb{N}$ such that

$$0 \leq \int_I (u(x) - s_N(x))\,\mathrm{d}x = \left| \int_I u(x)\,\mathrm{d}x - \int_I s_N(x)\,\mathrm{d}x \right| < \frac{\epsilon}{2}. \tag{26}$$

Let $s = s_N$ and $g = u - (v + s_N)$ on I. Clearly $s \in \mathcal{S}(I)$ and $g = \mathcal{L}(I)$. Also (a) follows immediately. On the other hand, we have

$$|g(x)| \leq |u(x) - s_N(x)| + |v(x)| = (u(x) - s_N(x)) + v(x)$$

for almost all $x \in I$. It follows from Theorem 4L, (25) and (26) that

$$\int_I |g(x)|\,\mathrm{d}x \leq \int_I (u(x) - s_N(x) + v(x))\,\mathrm{d}x = \int_I (u(x) - s_N(x))\,\mathrm{d}x + \int_I v(x)\,\mathrm{d}x < \epsilon.$$

This gives (b). ○

4.4. Sets of Measure Zero

In this section, we shall show that the behaviour of a Lebesgue integrable function on a set of measure zero does not affect the integral. More precisely, we prove the following result.

THEOREM 4T. *Suppose that $I \subseteq \mathbb{R}$ is an interval, and that $f \in \mathcal{L}(I)$. Suppose further that the function $g : I \to \mathbb{R}$ is such that $f(x) = g(x)$ for almost all $x \in I$. Then $g \in \mathcal{L}(I)$, and*

$$\int_I f(x)\,\mathrm{d}x = \int_I g(x)\,\mathrm{d}x.$$

EXAMPLE 4.4.1. Consider the function $g : [0,1] \to \mathbb{R}$, defined by

$$g(x) = \begin{cases} 0 & \text{if } x \text{ is rational,} \\ 1 & \text{if } x \text{ is irrational.} \end{cases}$$

Let $f(x) = 1$ for every $x \in [0,1]$. Then $f \in \mathcal{L}([0,1])$, and

$$\int_{[0,1]} f(x)\,\mathrm{d}x = 1.$$

Note next that the set of rational numbers in $[0,1]$ is a set of measure zero. It follows from Theorem 4T that $g \in \mathcal{L}([0,1])$, and

$$\int_{[0,1]} g(x)\,\mathrm{d}x = 1.$$

Recall, however, that the function g is not Riemann integrable over $[0,1]$.

The proof of Theorem 4T depends on the following intermediate result.

THEOREM 4U. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the function $f : I \to \mathbb{R}$ is such that $f(x) = 0$ for almost all $x \in I$. Then $f \in \mathcal{L}(I)$, and*

$$\int_I f(x)\,\mathrm{d}x = 0.$$

PROOF. Let $s_n : I \to \mathbb{R}$ satisfy $s_n(x) = 0$ for all $x \in I$. Then s_n is an increasing sequence of step functions which converges to 0 everywhere in I. It follows that $s_n(x) = f(x)$ for almost all $x \in I$. Furthermore, it is clear that

$$\lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x = 0.$$

It follows that $f \in \mathcal{U}(I)$, and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I s_n(x)\,\mathrm{d}x = 0$$

as required. ○

PROOF OF THEOREM 4T. In view of Theorem 4U, we have $g - f \in \mathcal{L}(I)$, and

$$\int_I (g(x) - f(x))\,\mathrm{d}x = 0.$$

Note next that $g = f + (g - f)$, and the result follows from Theorem 4L(a). ○

4.5. Relationship with Riemann Integration

We conclude this chapter by showing that Lebesgue integration is indeed a generalization of Riemann integration. We prove the following result. Suppose that $A, B \in \mathbb{R}$ and $A < B$ throughout this section.

THEOREM 4V. *Suppose that the function $f : [A, B] \to \mathbb{R}$ is bounded. Suppose further that f is Riemann integrable over $[A, B]$.*
(a) Then the set $\mathcal{D}$ of discontinuities of f in $[A, B]$ has measure zero.
(b) Furthermore, $f \in \mathcal{U}([A, B])$, and the Lebesgue integral of f over $[A, B]$ is equal to the Riemann integral of f over $[A, B]$.

REMARKS. (1) In fact, it can be shown that for any bounded function $f : [A, B] \to \mathbb{R}$, the condition (a) is equivalent to the condition that f is Riemann integrable over $[A, B]$.

(2) Note that if f is Riemann integrable over $[A, B]$, then it is an upper function on $[A, B]$. We shall show in the proof that the step functions generating f arise from some lower Riemann sums.

PROOF OF THEOREM 4V. (a) For every $x \in [A, B]$, write

$$\omega(x) = \lim_{h\to 0+} \sup_{y\in[A,B]\cap(x-h,x+h)} |f(y) - f(x)|.$$

It can be shown that $\omega(x_0) = 0$ if and only if f is continuous at x_0. It follows that we can write

$$\mathcal{D} = \bigcup_{k=1}^{\infty} \mathcal{D}_k,$$

where, for every $k \in \mathbb{N}$,

$$\mathcal{D}_k = \left\{ x \in [A,B] : \omega(x) \geq \frac{1}{k} \right\}.$$

Suppose on the contrary that $\mathcal{D}$ does not have measure zero. Then by Theorem 3M, there exists $k_0 \in \mathbb{N}$ such that $\mathcal{D}_{k_0}$ does not have measure zero, so that there exists $\epsilon_0 > 0$ such that every countable collection of open intervals covering $\mathcal{D}_{k_0}$ has a sum of lengths at least ϵ_0. Suppose that

$$\Delta: \; A = x_0 < x_1 < x_2 < \ldots < x_n = B$$

is a dissection of the interval $[A,B]$. Then

$$S(f,\Delta) - s(f,\Delta) = \sum_{i=1}^{n} (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1},x_i]} f(x) - \inf_{x \in [x_{i-1},x_i]} f(x) \right).$$

Write

$$S(f,\Delta) - s(f,\Delta) = T_1 + T_2 \geq T_1, \tag{27}$$

where

$$T_1 = \sum_{\substack{i=1 \\ (x_{i-1},x_i) \cap \mathcal{D}_{k_0} \neq \emptyset}}^{n} (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1},x_i]} f(x) - \inf_{x \in [x_{i-1},x_i]} f(x) \right) \tag{28}$$

and

$$T_2 = \sum_{\substack{i=1 \\ (x_{i-1},x_i) \cap \mathcal{D}_{k_0} = \emptyset}}^{n} (x_i - x_{i-1}) \left(\sup_{x \in [x_{i-1},x_i]} f(x) - \inf_{x \in [x_{i-1},x_i]} f(x) \right).$$

Note that the open intervals in T_1 cover $\mathcal{D}_{k_0}$, with the possible exception of a finite number of points (which has total measure zero). It follows that the total length of the intervals in T_1 is at least ϵ_0. In other words,

$$\sum_{\substack{i=1 \\ (x_{i-1},x_i) \cap \mathcal{D}_{k_0} \neq \emptyset}}^{n} (x_i - x_{i-1}) \geq \epsilon_0. \tag{29}$$

On the other hand,

$$\sup_{x \in [x_{i-1},x_i]} f(x) - \inf_{x \in [x_{i-1},x_i]} f(x) \geq \frac{1}{k_0} \tag{30}$$

whenever $(x_{i-1},x_i) \cap \mathcal{D}_{k_0} \neq \emptyset$. Combining (28)–(30), we conclude that

$$T_1 \geq \frac{\epsilon_0}{k_0}. \tag{31}$$

It now follows from (27) and (31) that

$$S(f,\Delta) - s(f,\Delta) \geq \frac{\epsilon_0}{k_0}. \tag{32}$$

Note finally that (32) holds for every dissection Δ of $[A,B]$. It follows that f is not Riemann integrable over $[A,B]$.

(b) For every $n \in \mathbb{N}$, consider the dissection

$$\Delta_n : \ A = x_0 < x_1 < x_2 < \ldots < x_{2^n} = B$$

of the interval $[A, B]$ into 2^n equal subintervals of length $(B - A)/2^n$, and note that the subintervals of Δ_{n+1} can be obtained by bisecting the subintervals of Δ_n. For every $i = 1, \ldots, 2^n$, let

$$m_i = \inf\{f(x) : x \in [x_{i-1}, x_i]\}, \tag{33}$$

and define a step function $s_n : [A, B] \to \mathbb{R}$ by

$$s_n(x) = \begin{cases} m_i & \text{if } x \in (x_{i-1}, x_i], \\ m_1 & \text{if } x = x_0. \end{cases} \tag{34}$$

It is easy to check (the reader is advised to draw a picture) that

$$s_n(x) \le f(x) \tag{35}$$

for every $x \in [A, B]$, and that the sequence s_n is increasing on $[A, B]$. To show that $f \in \mathcal{U}([A, B])$, it remains to show that $s_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in [A, B]$, and that the sequence

$$\int_{[A,B]} s_n(x) \, \mathrm{d}x = s(f, \Delta_n) \tag{36}$$

is convergent. Since the set of discontinuities of f in $[A, B]$ has measure zero, to show that $s_n(x) \to f(x)$ as $n \to \infty$ for almost all $x \in [A, B]$, it suffices to show that $s_n(x_0) \to f(x_0)$ as $n \to \infty$ at every point x_0 of continuity of f. Suppose now that f is continuous at x_0. Then given any $\epsilon > 0$, there exists $\delta > 0$ such that

$$f(x_0) - \epsilon < f(x) < f(x_0) + \epsilon \qquad \text{for every } x \in (x_0 - \delta, x_0 + \delta).$$

Let

$$m(\delta) = \inf\{f(x) : x \in (x_0 - \delta, x_0 + \delta)\}. \tag{37}$$

Then $f(x_0) - \epsilon \le m(\delta)$, and so

$$f(x_0) \le m(\delta) + \epsilon. \tag{38}$$

On the other hand, there clearly exists $N \in \mathbb{N}$ large enough such that an interval $[x_{i-1}, x_i]$ in the dissection Δ_N contains x_0 and lies inside $(x_0 - \delta, x_0 + \delta)$; in other words,

$$x_0 \in [x_{i-1}, x_i] \subset (x_0 - \delta, x_0 + \delta) \tag{39}$$

(the reader is advised to draw a picture). Then, in view of (33)–(35) and (37)–(39), we have

$$S_N(x_0) \le f(x_0) \le m(\delta) + \epsilon \le m_i + \epsilon = S_N(x_0) + \epsilon. \tag{40}$$

Since the sequence s_n is increasing on $[A, B]$, it follows from (35) and (40) that for every $n > N$, we have

$$s_n(x_0) \le f(x_0) \le S_N(x_0) + \epsilon \le S_n(x_0) + \epsilon.$$

Hence $|s_n(x_0) - f(x_0)| < \epsilon$ for every $n > N$, whence $s_n(x_0) \to f(x_0)$ as $n \to \infty$. Finally, note that the sequence (36) is increasing and bounded above. Clearly it converges to the Riemann integral of f over $[A, B]$. ○