INTRODUCTION TO LEBESGUE INTEGRATION

W W L CHEN

Chapter 6

DOMINATED CONVERGENCE THEOREM

6.1. Lebesgue's Theorem

In this section, we shall deduce the following result from the Monotone convergence theorem studied in the last chapter. The result below is usually considered the cornerstone of Lebesgue integration theory.

THEOREM 6A. (LEBESGUE'S THEOREM) *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of functions $f_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $f_n : I \to \mathbb{R}$ converges almost everywhere to a limit function $f : I \to \mathbb{R}$.
(b) There exists a non-negative function $F \in \mathcal{L}(I)$ such that for every $n \in \mathbb{N}$, $|f_n(x)| \le F(x)$ for almost all $x \in I$.
Then the limit function $f \in \mathcal{L}(I)$, the sequence

$$\int_I f_n(x)\,\mathrm{d}x$$

is convergent, and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x. \tag{1}$$

Remark. Note condition (b) that the sequence f_n is dominated by F almost everywhere.

Proof of Theorem 6A. We shall construct two sequences $g_n, h_n \in \mathcal{L}(I)$ such that

$$g_n(x) \le f_n(x) \le h_n(x) \tag{2}$$

for every $x \in I$, and where g_n is increasing and h_n is decreasing on I, and both converge to the limit function f almost everywhere on I. Clearly the sequence

$$\int_I g_n(x)\,\mathrm{d}x$$

is increasing and bounded above by

$$\int_I F(x)\,\mathrm{d}x,$$

so that

$$\lim_{n\to\infty}\int_I g_n(x)\,\mathrm{d}x$$

exists. It follows from Theorem 5C that $f \in \mathcal{L}(I)$ and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I g_n(x)\,\mathrm{d}x. \tag{3}$$

On the other hand, the sequence

$$\int_I h_n(x)\,\mathrm{d}x$$

is decreasing and bounded below by

$$-\int_I F(x)\,\mathrm{d}x,$$

so that

$$\lim_{n\to\infty}\int_I h_n(x)\,\mathrm{d}x$$

exists. It follows from Theorem 5C (applied to the sequence $-h_n$) that

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I h_n(x)\,\mathrm{d}x. \tag{4}$$

Combining (3) and (4), we obtain

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I g_n(x)\,\mathrm{d}x = \lim_{n\to\infty}\int_I h_n(x)\,\mathrm{d}x. \tag{5}$$

On the other hand, it follows from (2) that for every $n \in \mathbb{N}$,

$$\int_I g_n(x)\,\mathrm{d}x \le \int_I f_n(x)\,\mathrm{d}x \le \int_I h_n(x)\,\mathrm{d}x. \tag{6}$$

The equality (1) follows on letting $n \to \infty$ in (6) and combining with (5). It remains to establish the existence of the sequences g_n and h_n. For every $n \in \mathbb{N}$, write

$$h_n(x) = \sup\{f_n(x), f_{n+1}(x), f_{n+2}(x), \ldots\}$$

for every $x \in I$. Clearly $f_n(x) \le h_n(x)$ for every $x \in I$, and h_n is decreasing on I. Suppose that $x \in I$ and $f_n(x) \to f(x)$ as $n \to \infty$. Then given any $\epsilon > 0$, there exists N such that for every $n > N$,

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon.$$

It follows that for all $n > N$,

$$f(x) - \epsilon < h_n(x) \leq f(x) + \epsilon,$$

so that $h_n(x) \to f(x)$ as $n \to \infty$. Since $f_n \to f$ as $n \to \infty$ almost everywhere on I, it follows that $h_n \to f$ as $n \to \infty$ almost everywhere on I. Unfortunately, we also need to show that $h_n \in \mathcal{L}(I)$. Here, the difficulty arises since $h_n(x)$ is defined as the supremum of a collection which may be finite or infinite. This difficulty would not have arisen if the collection were finite, since the supremum of such a collection would then be equal to its maximum, and we could then use Theorem 4N repeatedly. However, the finite case suggests the following approach. For every $m, n \in \mathbb{N}$ with $m > n$, write

$$h_{nm}(x) = \max\{f_n(x), f_{n+1}(x), \ldots, f_m(x)\}$$

for every $x \in I$. Then by repeated application of Theorem 4N, we have $h_{nm} \in \mathcal{L}(I)$. For every fixed $n \in \mathbb{N}$, the sequence h_{nm} (in counting variable $m > n$) is increasing on I. On the other hand, clearly $|h_{nm}(x)| \leq F(x)$ for almost all $x \in I$. It follows that

$$\left| \int_I h_{nm}(x) \, \mathrm{d}x \right| \leq \int_I |h_{nm}(x)| \, \mathrm{d}x \leq \int_I F(x) \, \mathrm{d}x.$$

Hence the sequence

$$\int_I h_{nm}(x) \, \mathrm{d}x$$

is increasing and bounded above and so converges. It follows from Theorem 5C that h_{nm} converges almost everywhere as $m \to \infty$ to a limit function in $\mathcal{L}(I)$. Clearly $h_{nm} \to h_n$ as $m \to \infty$. This proves that $h_n \in \mathcal{L}(I)$. Similarly, write

$$g_n(x) = \inf\{f_n(x), f_{n+1}(x), f_{n+2}(x), \ldots\}$$

for every $x \in I$. Clearly $g_n(x) \leq f_n(x)$ for every $x \in I$, and g_n is increasing on I. Suppose that $x \in I$ and $f_n(x) \to f(x)$ as $n \to \infty$. Then given any $\epsilon > 0$, there exists N such that for every $n > N$,

$$f(x) - \epsilon < f_n(x) < f(x) + \epsilon.$$

It follows that for all $n > N$,

$$f(x) - \epsilon \leq g_n(x) < f(x) + \epsilon,$$

so that $g_n(x) \to f(x)$ as $n \to \infty$. Since $f_n \to f$ as $n \to \infty$ almost everywhere on I, it follows that $g_n \to f$ as $n \to \infty$ almost everywhere on I. To show that $g_n \in \mathcal{L}(I)$, for every $m, n \in \mathbb{N}$ with $m > n$, write

$$g_{nm}(x) = \min\{f_n(x), f_{n+1}(x), \ldots, f_m(x)\}$$

for every $x \in I$. Then by repeated application of Theorem 4N, we have $g_{nm} \in \mathcal{L}(I)$. For every fixed $n \in \mathbb{N}$, the sequence g_{nm} (in counting variable $m > n$) is decreasing on I. On the other hand, clearly $|g_{nm}(x)| \leq F(x)$ for almost all $x \in I$. It follows that

$$\left| \int_I g_{nm}(x) \, \mathrm{d}x \right| \leq \int_I |g_{nm}(x)| \, \mathrm{d}x \leq \int_I F(x) \, \mathrm{d}x.$$

Hence the sequence

$$\int_I g_{nm}(x) \, \mathrm{d}x$$

is decreasing and bounded below and so converges. It follows from Theorem 5C (applied to the sequence $-g_{nm}$) that g_{nm} converges almost everywhere as $m \to \infty$ to a limit function in $\mathcal{L}(I)$. Clearly $g_{nm} \to g_n$ as $m \to \infty$. This proves that $g_n \in \mathcal{L}(I)$. The proof of Theorem 6A is now complete. ○

The following version for a series can be deduced easily from Theorem 6A.

THEOREM 6B. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of functions $g_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) $\displaystyle\sum_{n=1}^{\infty} g_n$ converges almost everywhere on I to a sum function $g : I \to \mathbb{R}$.
(b) There exists a non-negative function $G \in \mathcal{L}(I)$ such that for every $N \in \mathbb{N}$, $\displaystyle\left|\sum_{n=1}^{N} g_n(x)\right| \le G(x)$ for almost all $x \in I$.
Then $g \in \mathcal{L}(I)$, the series

$$\sum_{n=1}^{\infty} \int_I g_n(x)\,\mathrm{d}x$$

converges, and

$$\int_I g(x)\,\mathrm{d}x = \int_I \sum_{n=1}^{\infty} g_n(x)\,\mathrm{d}x = \sum_{n=1}^{\infty} \int_I g_n(x)\,\mathrm{d}x.$$

6.2. Consequences of Lebesgue's Theorem

The following result is sometimes called the Bounded convergence theorem.

THEOREM 6C. *Suppose that $I \subseteq \mathbb{R}$ is a bounded interval. Suppose further that the sequence of functions $f_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $f_n : I \to \mathbb{R}$ converges almost everywhere to a limit function $f : I \to \mathbb{R}$.
(b) There exists $M \in \mathbb{R}$ such that for every $n \in \mathbb{N}$, $|f_n(x)| \le M$ for almost all $x \in I$.
Then the limit function $f \in \mathcal{L}(I)$, and

$$\int_I f(x)\,\mathrm{d}x = \lim_{n\to\infty} \int_I f_n(x)\,\mathrm{d}x.$$

Remark. In view of conditions (a) and (b), we say that the sequence f_n is boundedly convergent almost everywhere on I.

Proof of Theorem 6C. Let $F(x) = M$ for every $x \in I$, and note that since I is a bounded interval, we have $F \in \mathcal{L}(I)$. The result now follows from Theorem 6A. ○

The last result in this section is sometimes useful in establishing Lebesgue integrability.

THEOREM 6D. *Suppose that $I \subseteq \mathbb{R}$ is an interval. Suppose further that the sequence of functions $f_n \in \mathcal{L}(I)$ satisfies the following conditions:*
(a) The sequence $f_n : I \to \mathbb{R}$ converges almost everywhere to a limit function $f : I \to \mathbb{R}$.
(b) There exists a non-negative function $F \in \mathcal{L}(I)$ such that $|f(x)| \le F(x)$ for almost all $x \in I$.
Then the limit function $f \in \mathcal{L}(I)$.

Proof. For every $n \in \mathbb{N}$, write

$$g_n(x) = \max\{\min\{f_n(x), F(x)\}, -F(x)\}$$

for every $x \in I$ (the reader is advised to draw a picture). Then $g_n \in \mathcal{L}(I)$ by Theorem 4N. It is easy to see that $|g_n(x)| \le F(x)$ for almost all $x \in I$, and that $g_n \to f$ as $n \to \infty$ almost everywhere on I. The result follows from Theorem 6A. $\bigcirc$