CHAPTER 1

Introduction

1.1. A Special Case

In the simplest case, an ordinary differential equation of order k is an equation of the type

$$F(t, x, x', \ldots, x^{(k)}) = 0, \tag{1.1}$$

where $t \in \mathbb{R}$ is known as the independent variable, $x = x(t)$ is an unknown scalar function, and where F is a function defined on some subset of $\mathbb{R}^{k+2}$. We have assumed that the left hand side of (1.1) is dependent on the term $x^{(k)}$.

A function $x = \varphi(t)$, $r_1 < t < r_2$, which reduces the equation (1.1) to an identity, is called a solution of (1.1) in the interval (r_1, r_2).

Without loss of generality, we assume that $k \geqslant 1$. It then follows that $\varphi(t)$ is k-times differentiable and hence continuous on (r_1, r_2). Let $S \subseteq \mathbb{R}^{k+2}$ be the domain of the function F. Then

$$(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(k)}(t)) \in S, \quad t \in (r_1, r_2).$$

Suppose further that we can solve (locally) for $x^{(k)}$. Then

$$x^{(k)} = G(t, x, x', \ldots, x^{(k-1)}) \tag{1.2}$$

is called the k-th order equation in normal form. Clearly $G : \mathcal{B} \to \mathbb{R}$ for some subset $\mathcal{B} \subseteq \mathbb{R}^{k+1}$.

Example. The equation

$$F(t, x, x', x'') = (1 + t)x'' - xx' + t = 0,$$

where $x = x(t)$ is an unknown function, is an ordinary differential equation of order 2. Written in normal form, it becomes

$$x'' = G(t, x, x') = \frac{xx' - t}{1 + t},$$

where $G : \mathcal{B} \to \mathbb{R}$ for some subset $\mathcal{B} \subseteq \mathbb{R}^3$.

1.2. Another Special Case

Consider an equation of the type

$$F(t, x, x') = 0, \tag{1.3}$$

where again $t \in \mathbb{R}$ is the independent variable, $x = x(t) = (x_1(t), \ldots, x_n(t))$ is an unknown n-dimensional vector function,

$$x' = x'(t) = (x_1'(t), \ldots, x_n'(t)),$$

and where F is a function defined on some subset of $\mathbb{R}^{2n+1}$. Again we have assumed that the left hand side of (1.3) is dependent on the term x'.

A function $x = \varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$, $r_1 < t < r_2$, which reduces the equation (1.3) to an identity, is called a solution of (1.3) in the interval (r_1, r_2). Clearly $\varphi(t)$ is differentiable and hence continuous on (r_1, r_2).

Let $S \subseteq \mathbb{R}^{2n+1}$ be the domain of the function F. Then

$$(t, \varphi(t), \varphi'(t)) \in S, \quad t \in (r_1, r_2).$$

Again, suppose that we can solve (locally) for x'. Then

$$x' = G(t, x) \tag{1.4}$$

is in normal form. Clearly $G : \mathcal{B} \to \mathbb{R}^n$ for some subset $\mathcal{B} \subseteq \mathbb{R}^{n+1}$.

EXAMPLE. Let $n = 3$. Consider an equation of the type (1.3), where $x = x(t) = (x_1(t), \ldots, x_3(t))$ is an unknown 3-dimensional vector function that satisfies

$$\begin{aligned} x_1' &= \ tx_1 + 2x_2 + tx_3 + t, \\ x_2' &= \ 3x_1 + tx_2 + 4x_3 + t^2, \\ x_3' &= t^2 x_1 + tx_2 + 6x_3 + 2t. \end{aligned}$$

Then

$$G(t, x) = G(t, x_1, \ldots, x_3) = (tx_1 + 2x_2 + tx_3 + t, 3x_1 + tx_2 + 4x_3 + t^2, t^2 x_1 + tx_2 + 6x_3 + 2t),$$

where $G : \mathcal{B} \to \mathbb{R}^3$ for some subset $\mathcal{B} \subseteq \mathbb{R}^4$.

1.3. The General Case

The general form of a k-th order ordinary differential equation is

$$F(t, x, x', \ldots, x^{(k)}) = 0, \tag{1.5}$$

where $t \in \mathbb{R}$ is the independent variable, $x = x(t) = (x_1(t), \ldots, x_n(t))$ is an unknown n-dimensional vector function,

$$\begin{aligned} x' = x'(t) &= (x_1'(t), \ldots, x_n'(t)), \\ x'' = x''(t) &= (x_1''(t), \ldots, x_n''(t)), \\ &\vdots \\ x^{(k)} = x^{(k)}(t) &= (x_1^{(k)}(t), \ldots, x_n^{(k)}(t)), \end{aligned}$$

and where F is a function defined on some subset of $\mathbb{R}^{(k+1)n+1}$. We have assumed that the left hand side of (1.5) is dependent on the term $x^{(k)}$.

A function $x = \varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t))$, $r_1 < t < r_2$, which reduces the equation (1.5) to an identity, is called a solution of (1.5) in the interval (r_1, r_2).

Without loss of generality, we assume that $k \geqslant 1$. It then follows that $\varphi(t)$ is k-times differentiable and hence continuous on (r_1, r_2). Let $S \subseteq \mathbb{R}^{(k+1)n+1}$ be the domain of the function F. Then

$$(t, \varphi(t), \varphi'(t), \ldots, \varphi^{(k)}(t)) \in S, \quad t \in (r_1, r_2).$$

Suppose further that we can solve (locally) for $x^{(k)}$. Then

$$x^{(k)} = G(t, x, x', \ldots, x^{(k-1)}) \tag{1.6}$$

is called the k-th order equation in normal form. Clearly $G : \mathcal{B} \to \mathbb{R}^n$ for some subset $\mathcal{B} \subseteq \mathbb{R}^{kn+1}$.

An equation of the form (1.6) with $n > 1$ is sometimes called a k-th order n-dimensional system.

1.4. A Reduction Argument

Consider the case $n = 1$. Equation (1.1) is of the form

$$F(t, x, x', \ldots, x^{(k)}) = 0,$$

where $x = x(t)$ is an unknown scalar function, and where F is a function defined on some subset of $\mathbb{R}^{k+2}$.

Let $y_1 = x$, $y_2 = x'$, $y_3 = x''$, $\ldots$, $y_k = x^{(k-1)}$. Then

$$y_1' = y_2, \quad y_2' = y_3, \quad \ldots, \quad y_{k-1}' = y_k. \tag{1.7}$$

It follows that equation (1.2) is of the form

$$y_k' = x^{(k)} = G(t, x, x', \ldots, x^{(k-1)}) = G(t, y_1, y_2, \ldots, y_k). \tag{1.8}$$

Combining (1.7) and (1.8), we have

$$(y_1', y_2', \ldots, y_{k-1}', y_k') = (y_2, y_3, \ldots, y_k, G(t, y_1, y_2, \ldots, y_k)),$$

so that writing $y = (y_1, \ldots, y_k)$ and

$$f(t, y) = (y_2, y_3, \ldots, y_k, G(t, y_1, y_2, \ldots, y_k)),$$

we obtain

$$y' = f(t, y),$$

an equation of type (1.4) in terms of the variable t and the k-dimensional vector y.

In summary, we have reduced a k-th order ordinary differential equation with unknown scalar variable x to a first order k-dimensional system.

A similar, but notationally more complicated, argument will show that any equation of the type (1.5), *i.e.* a k-th order n-dimensional system, can be reduced to a first order kn-dimensional system. It follows that the study of ordinary differential equations is reduced to the study of first order systems.

Example. Consider a second order system

$$\begin{aligned} x_1'' &= ax_1' + bx_2', \\ x_2'' &= cx_1' + dx_2, \end{aligned}$$

where $x = x(t) = (x_1(t), x_2(t))$ is an unknown two-dimensional vector. Write $(y_1, y_2) = x = (x_1, x_2)$ and $(y_3, y_4) = x' = (x_1', x_2') = (y_1', y_2')$. Then

$$y'(t) = (y_1'(t), y_2'(t), y_3'(t), y_4'(t)) = (y_3(t), y_4(t), ay_3(t) + by_4(t), cy_3(t) + dy_2(t)) = f(t, y).$$

We now have a first order four-dimensional system.

Our study of ordinary differential equations will henceforth concentrate on first order equations of the type

$$x' = f(t, x), \tag{1.9}$$

where $x = x(t) = (x_1(t), \ldots, x_n(t))$ is an unknown n-dimensional vector function, and where the function $f : \mathcal{B} \to \mathbb{R}^n$ for some subset $\mathcal{B} \subseteq \mathbb{R}^{n+1}$.

1.5. Integral Curves and Direction Field

Consider an equation of the type (1.9), where $f(t, x)$ is defined on an open connected set $\mathcal{B} \subseteq \mathbb{R}^{n+1}$.

Suppose that (i) $f(t, x)$ is continuous on $\mathcal{B}$; and (ii) the function

$$x = \varphi(t) = (\varphi_1(t), \ldots, \varphi_n(t)), \quad t \in (r_1, r_2)), \tag{1.10}$$

is a solution of (1.9). If we think of the solution (1.10) as a curve lying entirely in $\mathcal{B}$, then

$$\varphi'(t) = (\varphi_1'(t), \ldots, \varphi_n'(t)) = f(t, \varphi(t)), \tag{1.11}$$

so that $\varphi'(t)$ is continuous on (r_1, r_2). It follows that the curve (1.10) has a continuously turning tangent at each point given by (1.11). Such a curve is called an integral curve.

Suppose now that $(t, x) \in \mathcal{B}$. Let $\phi_{t,x}$ be a line segment passing through the point (t, x) and parallel to the vector $f(t, x)$. Then

$$\{\phi_{t,x} : (t, x) \in \mathcal{B}\}$$

is called the direction field of the equation (1.9).

1.6. The Theorem

It has often been said that there is only one theorem in the theory of ordinary differential equations. We state a version of it as follows. The proof will be given later.

Theorem. *Consider the differential equation* (1.9)*, where the function* $f(t, x)$ *is defined on some domain* $\mathcal{B} \subseteq \mathbb{R}^{n+1}$*. Suppose further that*

(i) f *is continuous on* $\mathcal{B}$*; and*
(ii) *for every* $i = 1, \ldots, n$*, the function* $\partial f / \partial x_i$ *is defined and continuous on* $\mathcal{B}$*.*

Then for every point $(t_0, x_0) \in \mathcal{B}$*, there exists a unique solution* $x = \varphi(t)$ *of* (1.9) *satisfying* $x_0 = \varphi(t_0)$ *and defined in some neighbourhood of* (t_0, x_0)*.*

By uniqueness, we mean that if $x = \varphi(t)$ and $x = \psi(t)$ are both solutions of (1.9) satisfying $x_0 = \varphi(t_0) = \psi(t_0)$ and each is defined in some neighbourhood of (t_0, x_0), then the two solutions are identical in their common interval of definition.

We call the pair (t_0, x_0) the initial values for the solution $x = \varphi(t)$. The relation $x_0 = \varphi(t_0)$ is called the initial condition for the solution $x = \varphi(t)$.

EXAMPLE. Let $n = 1$. The functions $\varphi(t) = (t - t_0)^3$ and $\varphi(t) = 0$, with $t, t_0 \in (-\infty, \infty)$, are both solutions of the equation $x' = 3x^{2/3}$. Also both satisfy the initial condition $\varphi(t_0) = 0$. Note that while $f(t, x) = 3x^{2/3}$ is continuous at $x = 0$, $\partial f/\partial x = 2x^{-1/3}$ does not exist at $x = 0$. It can be checked that with any other initial condition $\varphi(t_0) = x_0$, where $x_0 \neq 0$, the hypotheses of the theorem are satisfied, and we have unique solution. Note that the initial values (t_0, x_0) satisfy the hypotheses of the theorem if and only if $x_0 \neq 0$.

1.7. Maximal Interval of Existence

Suppose that $\varphi_1(t)$, $t \in (r_1, r_2)$, and $\varphi_2(t)$, $t \in (s_1, s_2)$, are both solutions of the differential equation $x' = f(t, x)$ under the same initial condition, so that $\varphi_1(t_0) = \varphi_2(t_0) = x_0$. Suppose further that the differential equation satisfies the hypotheses of the Theorem. Then in a neighbourhood of (t_0, x_0), we have uniqueness of solution, so that

$$\varphi_1(t) = \varphi_2(t), \quad t \in (\max\{r_1, s_1\}, \min\{r_2, s_2\}). \tag{1.12}$$

Now define

$$\varphi(t) = \begin{cases} \varphi_1(t), & \text{if } r_1 < t < r_2, \\ \varphi_2(t), & \text{if } s_1 < t < s_2. \end{cases}$$

In view of (1.12), it is easy to see that

$$\varphi(t), \quad t \in (\min\{r_1, s_1\}, \max\{r_2, s_2\}),$$

is a solution of $x' = f(t, x)$ satisfying $\varphi(t_0) = x_0$.

The above is a special case of the following result.

PROPOSITION 1.1. *Under the hypotheses and in the notation of the Theorem, given any initial values (t_0, x_0), there exists a solution $\varphi(t)$ of (1.9) satisfying $\varphi(t_0) = x_0$ and defined on an interval (m_1, m_2) with the following property: If $\psi(t)$, $t \in (r_1, r_2)$, is another solution of (1.9) satisfying $\psi(t_0) = x_0$, then we must have $(r_1, r_2) \subseteq (m_1, m_2)$.*

The interval (m_1, m_2) in Proposition 1.1 is sometimes called the maximal interval of existence corresponding to the initial values (t_0, x_0).

PROOF OF PROPOSITION 1.1. Denote by $\mathcal{S}$ the set of all intervals of definition of solutions of (1.9) satisfying the given initial condition. By the Theorem, a solution exists, so that $\mathcal{S}$ is non-empty. Denote by $\mathcal{S}_1$ the set of all left endpoints of the intervals in $\mathcal{S}$, and denote by $\mathcal{S}_2$ the set of all right endpoints of the intervals in $\mathcal{S}$; and let $m_1 = \inf \mathcal{S}_1$ and $m_2 = \sup \mathcal{S}_2$. We now define $\varphi(t)$, $t \in (m_1, m_2)$, as follows: For any $t_1 \in (m_1, m_2)$, there exists an interval $I \in \mathcal{S}$ and a solution $\psi(t)$, $t \in I$, of (1.9) such that $\psi(t_0) = x_0$ and $t_1 \in I$. We now define $\varphi(t_1) = \psi(t_1)$. In view of uniqueness, $\varphi(t)$, $t \in (m_1, m_2)$, is well defined and is a solution. Clearly the interval (m_1, m_2) satisfies the maximal requirement. ○

EXAMPLE. Let $n = 1$. Consider the differential equation

$$x' = -3x^{4/3} \sin t,$$

where $x = x(t)$ is an unknown scalar function. It is easy to check that the hypotheses of the Theorem are satisfied with $\mathcal{B} = \mathbb{R}^2$. Clearly $x(t) = 0$ may be a solution; this is the case with the initial values $(t_0, 0)$, and $(-\infty, \infty)$ is clearly the maximal interval of existence. On the other hand, $x(t) = (c - \cos t)^{-3}$, where c is determined by initial values, may also be a solution. Indeed, if $|c| > 1$, then

$$x(t) = (c - \cos t)^{-3}, \quad t \in (-\infty, \infty),$$

is a solution. However, if $|c| \leqslant 1$, then this is clearly not the case, as $(c - \cos t)^{-3}$ is undefined for some $t \in \mathbb{R}$. For example, the solution satisfying $x(\pi/2) = 1/8$ is

$$x(t) = (2 - \cos t)^{-3}, \quad t \in (-\infty, \infty),$$

whereas the solution satisfying $x(\pi/2) = 8$ is

$$x(t) = (1/2 - \cos t)^{-3}, \quad t \in (\pi/3, 5\pi/3);$$

both intervals are maximal for their respective initial values.

Problems for Chapter 1

1. Write each of the following differential equations as a first order system in normal form:
   (i) $x^{(4)} + x^{(3)} \cos t - x'' + x^2 \sin t = 0$, where $x = x(t)$ is a scalar function.
   (ii) $u'' + u = v' \sin t$, $v'' + v = u' \cos t$, where $u = u(t)$ and $v = v(t)$ are scalar functions.

2. Consider the differential equation $x' = f(x)$, where $x = x(t)$ is an unknown scalar function, and where f and $\mathrm{d}f/\mathrm{d}x$ are continuous on

$$\mathcal{B} = \{(t, x) : -\infty < t < \infty,\ a < x < b\}.$$

Suppose that $f(x) \neq 0$ for every $x \in (a, b)$, and let

$$F(x) = \int_{x_0}^{x} \frac{1}{f(s)}\, \mathrm{d}s,$$

where $x_0 \in (a, b)$.
   (i) Show that $x(t) = G(t - c)$, where G is the inverse function of F and where c is a constant determined by initial conditions, is a solution of $x' = f(x)$.
   (ii) Explain why all solutions of $x' = f(x)$ are of the form described in (i).

3. Consider the differential equation $x' = f(t, x)$, where $x = x(t)$ is an unknown scalar function, and where f and $\partial f/\partial x$ are continuous on

$$\mathcal{B} = \{(t, x) : r_1 < t < r_2,\ a < x < b\}.$$

Suppose that

$$f(t, x) = g(t)h(x) \tag{1.13}$$

for every $(t, x) \in \mathcal{B}$, where $h(x) \neq 0$ for every $x \in (a, b)$. Let $u = \varphi(t)$ be a solution of $u' = g(t)$, and let $x = \psi(u)$ be a solution of $\mathrm{d}x/\mathrm{d}u = h(x)$.
   (i) Show that $x(t) = \psi(\varphi(t) - c)$, where c is a constant determined by initial conditions, is a solution of $x' = f(t, x)$.
   (ii) Explain why all solutions of $x' = f(t, x)$ are of the form described in (i).
   (iii) Suppose that the function $F(\alpha t, \alpha u) = F(t, u)$ for every $\alpha \neq 0$, and that $u = u(t)$ is an unknown scalar function. Show that each of the subsitutions $u = tx$ and $u = t/x$, where $x = x(t)$, transforms the equation $u' = F(t, u)$ into an equation of the type $x' = f(t, x)$, where $f(t, x)$ is of the form (1.13).

4. For each of the following differential equations, discuss possible choices for the domain $\mathcal{B}$ of the Theorem, and find all solutions of the equation:
   (i) $x' = t\mathrm{e}^t$
   (ii) $x' = t \log(t^2 - 1)$
   (iii) $x' = x^2 - 4$
   (iv) $x' = \sec x$
   (v) $x' = -(t + 1)xt^{-1}$
   (vi) $x' = t^3(x + 1)^{-2}$
   (vii) $x' = (x + t)t^{-1}$
   [*Hint*: Use the substitution $x = ut$.]
   (viii) $x' = t^{-1}(x - (x^2 + t^2)^{1/2})$
   [*Hint*: Use the substitution $x = ut$.]

5. Consider the differential equation $x' = (x^2 - 1)/xt$, where $x = x(t)$ is an unknown scalar function.
   (i) Discuss possible choices for the domain $\mathcal{B}$ of the Theorem.
   (ii) Describe all solutions of the equation.
   (iii) Find the solution satisfying the initial condition $x(1) = 1/\sqrt{2}$.
   (iv) What is the maximal interval of existence of the solution in (iii)?
   [*Hint*: Take note of your solution of (i).]