CHAPTER 2

First Order Linear Equations

2.1. Homogeneous Equations

Consider the first order differential equation

$$x' = Ax, \tag{2.1}$$

where $x = x(t)$ is an unknown scalar function, and where A is a constant.

Suppose that $\varphi(t)$ is a solution of (2.1). Then $\varphi'(t) = A\varphi(t)$, so that

$$\mathrm{e}^{-At}(\varphi'(t) - A\varphi(t)) = (\mathrm{e}^{-At}\varphi(t))' = 0,$$

whence $\mathrm{e}^{-At}\varphi(t) = \alpha$ for some constant α.

Suppose now that we have the initial condition $x(t_0) = x_0$. Then $\alpha = \mathrm{e}^{-At_0}\varphi(t_0) = \mathrm{e}^{-At_0}x_0$. It follows that

$$\varphi(t) = x_0\mathrm{e}^{A(t-t_0)}. \tag{2.2}$$

Note now that $\Omega(t) = \mathrm{e}^{A(t-t_0)}$ is a solution of (2.1) satisfying $\Omega(t_0) = 1$. Hence (2.2) can be rewritten in the form

$$\varphi(t) = \Omega(t)x_0. \tag{2.3}$$

We have proved

PROPOSITION 2.1. *Suppose that A is a constant. Then the solution of the differential equation* (2.1) *satisfying the initial condition $x(t_0) = x_0$ is given by* (2.3)*, where $\Omega(t)$ is a solution of* (2.1) *satisfying $\Omega(t_0) = 1$.*

Consider next the first order differential equation

$$x' = A(t)x, \tag{2.4}$$

where $x = x(t)$ is an unknown scalar function, and where the scalar function $A(t)$ is continuous on (r_1, r_2).

Suppose that $\varphi(t)$ is a solution of (2.4). Then $\varphi'(t) - A(t)\varphi(t) = 0$.

We shall find a function $u(t)$ such that

$$u(t)(\varphi'(t) - A(t)\varphi(t)) = (u(t)\varphi(t))'.$$

It is easily checked that an example of such a function is given by

$$\exp\left(-\int_{t_0}^{t} A(s)\,\mathrm{d}s\right). \tag{2.5}$$

Then $u(t)\varphi(t) = \alpha$ for some constant α.

Suppose now that we have the initial condition $x(t_0) = x_0$. Then $\alpha = u(t_0)\varphi(t_0) = u(t_0)x_0$. It follows that

$$\varphi(t) = \frac{u(t_0)}{u(t)}x_0. \tag{2.6}$$

Note now that $\Omega(t) = u(t_0)/u(t)$ is a solution of (2.4) satisfying $\Omega(t_0) = 1$. Hence (2.6) can be rewritten in the form

$$\varphi(t) = \Omega(t)x_0. \tag{2.7}$$

We have proved

PROPOSITION 2.2. *Suppose that $A(t)$ is continuous on (r_1, r_2). Suppose further that $t_0 \in (r_1, r_2)$. Then the solution of the equation* (2.4) *satisfying the initial condition $x(t_0) = x_0$ is given by* (2.7), *where $\Omega(t)$ is a solution of* (2.4) *satisfying $\Omega(t_0) = 1$.*

2.2. Non-Homogeneous Equations

Consider now the first order differential equation

$$x' = A(t)x + B(t), \tag{2.8}$$

where $x = x(t)$ is an unknown scalar function, and where the scalar functions $A(t)$ and $B(t)$ are continuous on (r_1, r_2).

Consider first the equation (2.4) with initial condition $x(t_0) = x_0$. By Proposition 2.2, there exists a solution $\Omega(t)$ such that $\Omega(t_0) = 1$ and $x(t) = \Omega(t)x_0$ is a solution of (2.4) with $x(t_0) = x_0$. Using this solution $\Omega(t)$, we shall attempt to find a solution of (2.8) which satisfies the initial condition $x(t_0) = x_0$.

Consider the function

$$\varphi(t) = \Omega(t)c(t), \tag{2.9}$$

where we shall attempt to choose $c(t)$ so that (2.9) is a solution of (2.8) with $\varphi(t_0) = x_0$. Clearly we need

$$\Omega'(t)c(t) + \Omega(t)c'(t) = \varphi'(t) = A(t)\varphi(t) + B(t) = A(t)\Omega(t)c(t) + B(t). \tag{2.10}$$

Since $\Omega(t)$ is a solution of (2.4), we must have $\Omega'(t) = A(t)\Omega(t)$. For (2.10) to hold, we must therefore have

$$\Omega(t)c'(t) = B(t). \tag{2.11}$$

We note now that $\Omega(t) \neq 0$ for all $t \in (r_1, r_2)$. To see this, note that if $\Omega(t) = 0$ for some $t \in (r_1, r_2)$, then the function $\psi(t) = 0$, $t \in (r_1, r_2)$, is clearly a solution of (2.4), so that $\Omega(t) = \psi(t)$, $t \in (r_1, r_2)$, in view of uniqueness. We can therefore rewrite (2.11) in the form

$$c'(t) = \Omega^{-1}(t)B(t). \tag{2.12}$$

Let us return to (2.9), and ensure that the initial condition $\varphi(t_0) = x_0$ is satisfied. We therefore also need

$$x_0 = \varphi(t_0) = \Omega(t_0)c(t_0) = c(t_0). \tag{2.13}$$

We can now integrate (2.12) with the initial condition (2.13) to obtain

$$c(t) = x_0 + \int_{t_0}^{t} \Omega^{-1}(s)B(s)\,\mathrm{d}s.$$

We have thus proved

PROPOSITION 2.3. *Suppose that $A(t)$ and $B(t)$ are continuous on (r_1, r_2). Suppose further that $t_0 \in (r_1, r_2)$. Then the solution of the equation* (2.8) *with initial condition $x(t_0) = x_0$ is given by*

$$x(t) = \Omega(t)x_0 + \int_{t_0}^{t} \Omega(t)\Omega^{-1}(s)B(s)\,\mathrm{d}s,$$

where $\Omega(t)$ is a solution of (2.4) *satisfying $\Omega(t_0) = 1$.*

Recall that $\Omega(t) = u(t_0)/u(t)$, where $u(t)$ is given by (2.5), is a solution of (2.4) satisfying $\Omega(t_0) = 1$. In this case,

$$\Omega(t) = \exp\left(\int_{t_0}^{t} A(s)\,\mathrm{d}s\right).$$

It now follows from Proposition 2.3 that

$$x(t) = x_0 \exp\left(\int_{t_0}^{t} A(s)\,\mathrm{d}s\right) + \exp\left(\int_{t_0}^{t} A(s)\,\mathrm{d}s\right)\int_{t_0}^{t} \exp\left(-\int_{t_0}^{s} A(u)\,\mathrm{d}u\right) B(s)\,\mathrm{d}s,$$

a more familiar form of solution of the equation (2.8).

Problems for Chapter 2

1. Solve each of the following differential equations for the unknown scalar function $x = x(t)$ with given initial condition:

(i) $x' = x + 2\mathrm{e}^t$ and $x(0) = 1$
(ii) $x' = -t^{-1}x - t^{-2}$ and $x(1) = 1$
(iii) $x' = -(t+1)t^{-1}x - 3t^2\mathrm{e}^{-t}$ and $x(1) = 1$

2. The differential equation

$$y' + \alpha(t)y = \beta(t)y^k,$$

where k is a constant, is called the Bernoulli equation.

(i) Show that the substitution $x = y^{1-k}$ transforms this equation into a linear equation.
(ii) Find all the solutions of $y' - 2ty = ty^2$.

3. Consider the homogeneous differential equation $x' = A(t)x$, where the function $A(t)$ is continuous on $(-\infty, \infty)$ and periodic with period $\xi > 0$, *i.e.* $A(t+\xi) = A(t)$ for every $t \in (-\infty, \infty)$.

(i) Suppose that $\varphi(t)$ is a non-trivial solution, and that $\psi(t) = \varphi(t+\xi)$ for every $t \in (-\infty, \infty)$. Show that $\psi(t)$ is also a solution.
(ii) Show that there is a constant c such that $\varphi(t+\xi) = c\varphi(t)$ for every $t \in (-\infty, \infty)$. Find an explicit expression for this constant c.
(iii) What condition must the function $A(t)$ satisfy in order that there exists a non-trivial solution of period 2ξ?
(iv) If $A(t)$ is real valued, what is the condition in (iii)?
(v) If $A(t)$ is constant, what is the condition in (iii)?

4. Consider the non-homogeneous differential equation $x' = A(t)x + B(t)$, where the functions $A(t)$ and $B(t)$ are continuous and real valued on $(-\infty, \infty)$ and periodic with period $\xi > 0$, and that the function $B(t)$ is not identically zero.

(i) Show that a solution $\varphi(t)$ is periodic with period ξ if and only if $\varphi(0) = \varphi(\xi)$.
(ii) Show that there exists a unique solution with period ξ if there is no non-trivial solution of the homogeneous equation with period ξ.
(iii) Suppose that there is a non-trivial solution of the homegeneous equation with period ξ. Show that there are periodic solutions of period ξ of the non-homogeneous equation if and only if

$$\int_0^\xi \exp\left(-\int_0^t A(s)\,\mathrm{d}s\right) B(t)\,\mathrm{d}t = 0.$$

(iv) Find solutions of period 2π for the equation $x' = 3x + \sin t$.