CHAPTER 3

First Order Linear Systems

3.1. Matrix Formulation

We shall discuss a special type of first order n-dimensional system given by

$$x_i' = \sum_{j=1}^{n} a_{ij}(t)x_j + b_i(t), \quad i = 1, \ldots, n, \tag{3.1}$$

where $x(t) = (x_1(t), \ldots, x_n(t)$ is an unknown n-dimensional vector function and where, for $i, j = 1, \ldots, n$, $a_{ij}(t)$ and $b_i(t)$ are given scalar functions, continuous on (r_1, r_2).

Let $A(t) = (a_{ij}(t))_{n\times n}$ represent the $n \times n$ matrix of the scalar functions $a_{ij}(t)$, and let $B(t) = (b_1(t), \ldots, b_n(t))$. If we think of $x' = x'(t)$, $x = x(t)$ and $B(t)$ as n-dimensional column vectors, then the system (3.1) can be expressed in matrix form as

$$x' = A(t)x + B(t). \tag{3.2}$$

Consider now the vector function $f(t, x) = A(t)x+B(t)$. It is easy to see that with $\mathcal{B} = (r_1, r_2)\times\mathbb{R}^n$, the hypotheses of the Theorem are satisfied. It follows that given any initial values in $\mathcal{B}$, the system (3.2), and hence (3.1), has unique solution.

3.2. Homogeneous Systems

Suppose that $B(t) = 0$ identically. Then the system (3.2) becomes

$$x' = A(t)x. \tag{3.3}$$

We shall assume for the remainder of this chapter that $A(t)$ is continuous on (r_1, r_2).

Suppose that $\psi(t) = (\psi_1(t), \ldots, \psi_n(t))$ and $\xi(t) = (\xi_1(t), \ldots, \xi_n(t))$ are both solutions of (3.3). For every scalar constants α and β, consider now the function $\varphi(t) = \alpha\psi(t) + \beta\xi(t)$. Note that

$$\varphi'(t) = \alpha\psi'(t) + \beta\xi'(t) = \alpha A(t)\psi(t) + \beta A(t)\xi(t) = A(t)(\alpha\psi(t) + \beta\xi(t)) = A(t)\varphi(t),$$

so that $\varphi(t)$ is also a solution of (3.3). Repeating this argument a finite number of times if necessary, it can be shown that any finite linear combination of solutions of (3.3) is also a solution of (3.3). In other words, the system (3.3) is linear.

A simple consequence of the Theorem is the following. The proof is almost trivial.

Proposition 3.1. *Suppose that $A(t)$ is continuous on (r_1, r_2). Suppose further that $t_0 \in (r_1, r_2)$ and that $\varphi(t)$ is a solution of (3.3) satisfying $\varphi(t_0) = 0$. Then $\varphi(t) = 0$ for every $t \in (r_1, r_2)$.*

This linearity also leads us to ideas in linear algebra.

Definition. A collection of functions $\xi_1(t), \ldots, \xi_m(t)$ is linearly dependent over the interval (r_1, r_2) if there exist constants $\alpha_1, \ldots, \alpha_m$, not all zero, such that

$$\sum_{k=1}^{m} \alpha_k\xi_k(t) = 0$$

for every $t \in (r_1, r_2)$. A collection of functions is linearly independent over the interval (r_1, r_2) if it is not linearly dependent over the interval (r_1, r_2).

EXAMPLES. (1) The functions $\sin 2t$ and $\sin t \cos t$ are linearly dependent over any interval (r_1, r_2), for $\sin 2t - 2 \sin t \cos t = 0$ for every $t \in (r_1, r_2)$.

(2) The collection $1, t, t^2, \ldots, t^m$ of scalar functions is linearly independent over $(-\infty, \infty)$, for any linear combination

$$\sum_{k=0}^{m} \alpha_k t^k$$

is a polynomial, and has only finitely many zeros, unless $\alpha_0 = \alpha_1 = \ldots = \alpha_m = 0$.

(3) For every $i = 1, \ldots, n$, write

$$\chi_i(t) = (\underbrace{0, \ldots, 0}_{i-1}, 1, \underbrace{0, \ldots, 0}_{n-i}).$$

Then the collection of functions $\chi_1(t), \ldots, \chi_n(t)$ is linearly independent, for

$$\sum_{i=1}^{n} \alpha_i \chi_i(t) = (\alpha_1, \ldots, \alpha_n) = 0$$

if and only if $\alpha_1 = \ldots = \alpha_n = 0$.

PROPOSITION 3.2. *Suppose that $A(t)$ is continuous on (r_1, r_2), and $\varphi_1(t), \ldots, \varphi_m(t)$, $t \in (r_1, r_2)$, are solutions of* (3.3)*, linearly independent over (r_1, r_2). Then*

$$\varphi(t) = \sum_{k=1}^{m} \alpha_k \varphi_k(t) \neq 0, \quad t \in (r_1, r_2),$$

unless $\alpha_1 = \ldots = \alpha_m = 0$.

PROOF. By linearity, $\varphi(t)$ is a solution of (3.3). Suppose that $t_0 \in (r_1, r_2)$ and $\varphi(t_0) = 0$. Then by Proposition 3.1, we must have $\varphi(t) = 0$ for every $t \in (r_1, r_2)$, so that either $\alpha_1 = \ldots = \alpha_m = 0$, or $\varphi_1(t), \ldots, \varphi_m(t)$ are linearly dependent over (r_1, r_2). ◯

Using the idea of linear independence, we now attempt to describe the solutions of equations of the form (3.3).

DEFINITION. Suppose that

$$\varphi_1(t), \ldots, \varphi_n(t), \quad t \in (r_1, r_2), \tag{3.4}$$

are solutions of the first order n-dimensional linear system (3.3), linearly independent over (r_1, r_2). Then we say that (3.4) is a fundamental system of solutions of the system (3.3).

We shall show that any solution of the system (3.3) can be described in terms of this fundamental system of solutions of (3.3). However, before we do so, we must show that such a fundamental system of solutions of (3.3) exists.

PROPOSITION 3.3. *For every first order n-dimensional linear system* (3.3)*, where $A(t)$ is continuous on (r_1, r_2), a fundamental system of solutions exists.*

PROOF. For every $k = 1, \ldots, n$, write

$$e_k = (\underbrace{0, \ldots, 0}_{k-1}, 1, \underbrace{0, \ldots, 0}_{n-k}).$$

Then the vectors $e_1, \ldots, e_n$ are linearly independent. For any $t_0 \in (r_1, r_2)$ and any $k = 1, \ldots, n$, let $\varphi_k(t)$ denote the unique solution of (3.3) satisfying $\varphi_k(t_0) = e_k$. It remains to show that the solutions $\varphi_1(t), \ldots, \varphi_n(t)$ are linearly independent over (r_1, r_2). Suppose that

$$\varphi(t) = \sum_{k=1}^{n} \alpha_k \varphi_k(t)$$

is identically zero over (r_1, r_2). Then in particular,

$$\varphi(t_0) = \sum_{k=1}^{n} \alpha_k \varphi_k(t_0) = \sum_{k=1}^{n} \alpha_k e_k = (\alpha_1, \ldots, \alpha_n) = 0,$$

so that $\alpha_1 = \ldots = \alpha_n = 0$. ◯

PROPOSITION 3.4. *Every solution of a first order n-dimensional linear system* (3.3)*, where $A(t)$ is continuous on (r_1, r_2), is a linear combination of members of a fundamental system of solutions of* (3.3).

SKETCH OF PROOF. Suppose that $x(t)$, $t \in (r_1, r_2)$, is a solution of (3.3) with $x(t_0) = x_0 = (x_{10}, \ldots, x_{n0})$. Let $\varphi_1(t), \ldots, \varphi_n(t)$, $t \in (r_1, r_2)$, be the fundamental system of solutions of (3.3) in the proof of Proposition 3.3, and let

$$\varphi(t) = \sum_{k=1}^{n} x_{k0} \varphi_k(t), \quad t \in (r_1, r_2).$$

By linearity, $\varphi(t)$ is a solution of (3.3). It is easy to check that $\varphi(t_0) = x_0$. It follows from uniqueness that $x(t) = \varphi(t)$, $t \in (r_1, r_2)$. If our fundamental system of solutions is different from that in the proof of Proposition 3.3, then a slightly longer argument applies. ◯

3.3. The Wronskian

Consider again the linear system (3.3), where $A(t)$ is continuous on (r_1, r_2).

Suppose that $\varphi_1(t), \ldots, \varphi_n(t)$, $t \in (r_1, r_2)$, are solutions of (3.3). Whether these solutions will form a fundamental system of solutions of (3.3) depends on whether they are linearly independent over (r_1, r_2).

Suppose that

$$\varphi_j(t) = (\varphi_{1j}(t), \ldots, \varphi_{nj}(t)), \quad j = 1, \ldots, n.$$

Then the determinant

$$W(t) = \det \begin{pmatrix} \varphi_{11}(t) & \ldots & \varphi_{1n}(t) \\ \vdots & & \vdots \\ \varphi_{n1}(t) & \ldots & \varphi_{nn}(t) \end{pmatrix} \tag{3.5}$$

is called the Wronskian of the solutions $\varphi_1(t), \ldots, \varphi_n(t)$. If $\varphi_1(t), \ldots, \varphi_n(t)$ is a fundamental system of solutions of (3.3), then the matrix

$$\mathcal{W}(t) = \begin{pmatrix} \varphi_{11}(t) & \ldots & \varphi_{1n}(t) \\ \vdots & & \vdots \\ \varphi_{n1}(t) & \ldots & \varphi_{nn}(t) \end{pmatrix} \tag{3.6}$$

is called a fundamental matrix.

Our aim is to show that a matrix $\mathcal{W}(t)$ is a fundamental matrix if and only if the corresponding Wronskian $W(t)$ never vanishes. However, we shall first find a simple way for calculating the Wronskian. One such formula is given below.

PROPOSITION 3.5 (Liouville). *Suppose that $A(t)$ is continuous on (r_1, r_2), that $\varphi_1(t), \ldots, \varphi_n(t)$ are solutions of* (3.3)*, and that $t_0 \in (r_1, r_2)$. Then the Wronskian of $\varphi_1(t), \ldots, \varphi_n(t)$ is given by*

$$W(t) = W(t_0) \exp\left(\int_{t_0}^{t} \operatorname{tr} A(s) \, \mathrm{d}s \right), \quad t \in (r_1, r_2).$$

PROOF. Let i and j be fixed. Expanding by the i-th row, we have

$$W(t) = \det \mathcal{W}(t) = (-1)^{i+j} \varphi_{ij}(t) \det \mathcal{W}_{ij}(t) + \sum_{\substack{k=1 \\ k \neq j}}^{n} (-1)^{i+k} \varphi_{ik}(t) \det \mathcal{W}_{ik}(t), \tag{3.7}$$

where for every $i,k=1,\ldots,n$, the matrix $\mathcal{W}_{ik}(t)$ is obtained from the matrix $\mathcal{W}(t)$ by deleting the i-th row and k-th column. On the other hand, using the chain rule, we have

$$W'(t)=\sum_{i=1}^{n}\sum_{j=1}^{n}\frac{\partial W}{\partial\varphi_{ij}}(t)\varphi'_{ij}(t). \tag{3.8}$$

For any fixed pair i and j, note that $\mathcal{W}_{ij}(t)$ as well as all the terms corresponding to $k\neq j$ on the right hand side of (3.7) are independent of $\varphi_{ij}(t)$, so that

$$\frac{\partial W}{\partial\varphi_{ij}}(t)=(-1)^{i+j}\det\mathcal{W}_{ij}(t), \tag{3.9}$$

and so on combining (3.8) and (3.9), we obtain

$$W'(t)=\sum_{i=1}^{n}\left(\sum_{j=1}^{n}\varphi'_{ij}(t)(-1)^{i+j}\det\mathcal{W}_{ij}(t)\right)=\sum_{i=1}^{n}W_i(t), \tag{3.10}$$

where, for every $i=1,\ldots,n$,

$$W_i(t)=\sum_{j=1}^{n}\varphi'_{ij}(t)(-1)^{i+j}\det\mathcal{W}_{ij}(t)=\det\begin{pmatrix}\varphi_{11}(t) & \ldots & \varphi_{1n}(t)\\ \vdots & & \vdots\\ \varphi'_{i1}(t) & \ldots & \varphi'_{in}(t)\\ \vdots & & \vdots\\ \varphi_{n1}(t) & \ldots & \varphi_{nn}(t)\end{pmatrix}. \tag{3.11}$$

Note that the matrix on the right hand side is obtained from $\mathcal{W}(t)$ by replacing the i-th row by the derivatives of its entries. Note also that for every $j=1,\ldots,n$, $\varphi_j(t)=(\varphi_{1j}(t),\ldots,\varphi_{nj}(t))$ is a solution of (3.3), so that

$$\varphi'_{ij}(t)=\sum_{k=1}^{n}a_{ik}(t)\varphi_{kj}(t). \tag{3.12}$$

Combining (3.10)–(3.12), we obtain

$$\begin{aligned}W'(t)&=\sum_{i=1}^{n}\det\begin{pmatrix}\varphi_{11}(t) & \ldots & \varphi_{1n}(t)\\ \vdots & & \vdots\\ \sum\limits_{k=1}^{n}a_{ik}(t)\varphi_{k1}(t) & \ldots & \sum\limits_{k=1}^{n}a_{ik}(t)\varphi_{kn}(t)\\ \vdots & & \vdots\\ \varphi_{n1}(t) & \ldots & \varphi_{nn}(t)\end{pmatrix}\\ &=\sum_{i=1}^{n}\det\begin{pmatrix}\varphi_{11}(t) & \ldots & \varphi_{1n}(t)\\ \vdots & & \vdots\\ a_{ii}(t)\varphi_{i1}(t) & \ldots & a_{ii}(t)\varphi_{in}(t)\\ \vdots & & \vdots\\ \varphi_{n1}(t) & \ldots & \varphi_{nn}(t)\end{pmatrix}.\end{aligned}$$

Note that in the last step, we have multiplied, for each $k\neq i$, the k-th row by $-a_{ik}(t)$ and added it to the i-th row. It now follows that

$$W'(t)=\sum_{i=1}^{n}a_{ii}(t)W(t)=(\operatorname{tr}A(t))W(t). \tag{3.13}$$

The desired result follows on integrating (3.13) with respect to t. ○

PROPOSITION 3.6. *Suppose that $A(t)$ is continuous on (r_1,r_2). Suppose further that $\varphi_1(t),\ldots,\varphi_n(t)$ are solutions of (3.3). Then $\varphi_1(t),\ldots,\varphi_n(t)$ is a fundamental system of solutions of (3.3) if and only if the Wronskian $W(t)\neq 0$ for every $t\in(r_1,r_2)$.*

PROOF. ($\Rightarrow$) Suppose that $\varphi_1(t),\ldots,\varphi_n(t)$ is a fundamental system of solutions of (3.3). Then by Proposition 3.4, any non-trivial solution $\varphi(t)$ of (3.3) can be expressed in the form

$$\varphi(t)=\sum_{k=1}^{n}\alpha_k\varphi_k(t),$$

where $\alpha_1,\ldots,\alpha_n\in\mathbb{R}$ are not all zero. Furthermore, $\alpha_1,\ldots,\alpha_n$ are unique (why?). Suppose that $\alpha=(\alpha_1,\ldots,\alpha_n)$. Then we can write

$$\varphi(t)=\mathcal{W}(t)\alpha, \tag{3.14}$$

where $\varphi(t)$ and α are interpreted as n-dimensional column vectors. For every $t\in(r_1,r_2)$, the system (3.14) can be interpreted as a system of n linear equations in the n unknowns $\alpha_1,\ldots,\alpha_n$, and having unique solution, so that we must have $W(t)=\det\mathcal{W}(t)\neq 0$.

($\Leftarrow$) Suppose that $W(t)\neq 0$ for any $t\in(r_1,r_2)$. It follows that the columns of the matrix $\mathcal{W}(t)$ in (3.6) are linearly independent over (r_1,r_2). Note that the columns of $\mathcal{W}(t)$ represent solutions $\varphi_1(t),\ldots,\varphi_n(t)$ of (3.3). It follows that $\varphi_1(t),\ldots,\varphi_n(t)$ form a fundamental system of solutions of (3.3). ○

We now attempt to express our result in matrix notation.

PROPOSITION 3.7. *Suppose that $A(t)$ is continuous on (r_1,r_2). Then*

(i) *any fundamental matrix $\mathcal{W}(t)$ of (3.3) is a solution of the matrix differential equation $\Phi'(t)=A(t)\Phi(t)$;*
(ii) *the solution $x(t)$ of (3.3) satisfying the initial condition $x(t_0)=x_0$ is given by $x(t)=\mathcal{W}(t)\mathcal{W}^{-1}(t_0)x_0$; and*
(iii) *the matrix $\Omega(t)=\mathcal{W}(t)\mathcal{W}^{-1}(t_0)$ is a fundamental matrix of (3.3) satisfying $\Omega(t_0)=I$.*

PROOF. (i) Note that the j-th columns of $\mathcal{W}'(t)$ and $\mathcal{W}(t)$ are respectively $\varphi_j'(t)$ and $\varphi_j(t)$, and that $\varphi_j'(t)=A(t)\varphi_j(t)$.

(ii) Note that by Proposition 3.4, the solution $x(t)$ is a linear combination of the columns of $\mathcal{W}(t)$, and can therefore be expressed in the form

$$x(t)=\mathcal{W}(t)\alpha, \tag{3.15}$$

where $x(t)$ and $\alpha=(\alpha_1,\ldots,\alpha_n)$ are interpreted as n-dimensional column vectors. Note now that $\det\mathcal{W}(t_0)=W(t_0)\neq 0$, so that $\mathcal{W}^{-1}(t_0)$ exists. It follows from (3.15) with $t=t_0$ that

$$\alpha=\mathcal{W}^{-1}(t_0)x(t_0). \tag{3.16}$$

The result follows on combining (3.15) and (3.16).

(iii) It now follows from $\det\mathcal{W}^{-1}(t_0)\neq 0$ that the columns of

$$\Omega(t)=\mathcal{W}(t)\mathcal{W}^{-1}(t_0)$$

are linearly independent over (r_1,r_2). Also, the columns of $\Omega(t)$ are linear combinations of the columns of $\mathcal{W}(t)$ and are therefore solutions of (3.3). Hence $\Omega(t)$ is a fundamental matrix. Clearly $\Omega(t_0)=I$. ○

EXAMPLE. Consider the two-dimensional system $x'=2x+3y$ and $y'=x+4y$. Then clearly $A(t)=\begin{pmatrix}2&3\\1&4\end{pmatrix}$ and $\operatorname{tr}A(t)=2+4=6$. It is not difficult to see that the hypotheses of the Theorem are satisfied with $\mathcal{B}=\mathbb{R}^3$, so that all solutions $\varphi(t)$ are defined for $t\in(-\infty,\infty)$. Using $t_0=0$ and Proposition 3.5, the Wronskian of any fundamental system of solutions is given by

$$W(t)=W(0)\exp\left(\int_0^t 6\,\mathrm{d}s\right)=W(0)\mathrm{e}^{6t}.$$

Now a fundamental system of solutions is given by

$$\varphi_1(t)=(3\mathrm{e}^t,-\mathrm{e}^t)\quad\text{and}\quad\varphi_2(t)=(\mathrm{e}^{5t},\mathrm{e}^{5t}),$$

with

$$\mathcal{W}(t)=\begin{pmatrix}3\mathrm{e}^t&\mathrm{e}^{5t}\\-\mathrm{e}^t&\mathrm{e}^{5t}\end{pmatrix}.$$

Hence $W(0) = \det\begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix} = 4$, so that $W(t) = 4\mathrm{e}^{6t}$. A very simple calculation then gives $\mathcal{W}^{-1}(0) = \begin{pmatrix} 1/4 & -1/4 \\ 1/4 & 3/4 \end{pmatrix}$, so that

$$\Omega(t) = \frac{1}{4}\begin{pmatrix} 3\mathrm{e}^t & \mathrm{e}^{5t} \\ -\mathrm{e}^t & \mathrm{e}^{5t} \end{pmatrix}\begin{pmatrix} 1 & -1 \\ 1 & 3 \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 3\mathrm{e}^t + \mathrm{e}^{5t} & -3\mathrm{e}^t + 3\mathrm{e}^{5t} \\ -\mathrm{e}^t + \mathrm{e}^{5t} & \mathrm{e}^t + 3\mathrm{e}^{5t} \end{pmatrix}.$$

It is easily checked that $\Omega(0) = I$.

3.4. Non-Homogeneous Systems

We now use our knowledge on the homogeneous system

$$x' = A(t)x \tag{3.17}$$

to study the non-homogeneous system

$$x' = A(t)x + B(t). \tag{3.18}$$

Here again, we assume throughout that the matrix $A(t)$ and the vector $B(t) = (b_1(t), \ldots, b_n(t))$ are continuous on (r_1, r_2), so that solutions exist and are unique in $\mathcal{B} = (r_1, r_2) \times \mathbb{R}^n$, and that every such solution is defined on (r_1, r_2).

Consider the system (3.17) with initial condition $x(t_0) = x_0$. By Proposition 3.7, there exists a fundamental matrix $\Omega(t)$ such that $\Omega(t_0) = I$ and that $x(t) = \Omega(t)x_0$ is the solution of (3.17) with $x(t_0) = x_0$. Using this particular fundamental matrix $\Omega(t)$, we shall attempt to find a solution of (3.18) which satisfies the initial condition $x(t_0) = x_0$. This will then be the unique solution.

Consider the function

$$\varphi(t) = \Omega(t)c(t), \tag{3.19}$$

where we shall attempt to choose $c(t) = (c_1(t), \ldots, c_n(t))$ so that (3.19) is a solution of (3.18) with $\varphi(t_0) = x_0$. Clearly we need

$$\Omega'(t)c(t) + \Omega(t)c'(t) = \varphi'(t) = A(t)\varphi(t) + B(t) = A(t)\Omega(t)c(t) + B(t). \tag{3.20}$$

Since $\Omega(t)$ is a fundamental matrix of (3.17), it follows from Proposition 3.7(i) that $\Omega'(t) = A(t)\Omega(t)$. For (3.20) to hold, we must therefore have

$$\Omega(t)c'(t) = B(t). \tag{3.21}$$

On the other hand, in view of Proposition 3.6, $\det\Omega(t) \neq 0$ for all $t \in (r_1, r_2)$, so that $\Omega^{-1}(t)$ exists for every $t \in (r_1, r_2)$. Equation (3.21) can therefore be rewritten in the form

$$c'(t) = \Omega^{-1}(t)B(t). \tag{3.22}$$

Let us return to (3.19), and ensure that the initial condition $\varphi(t_0) = x_0$ is satisfied. We therefore also need

$$x_0 = \varphi(t_0) = \Omega(t_0)c(t_0) = Ic(t_0) = c(t_0). \tag{3.23}$$

We now integrate (3.22) with the initial condition (3.23) to obtain

$$c(t) = x_0 + \int_{t_0}^{t} \Omega^{-1}(s)B(s)\,\mathrm{d}s.$$

Here, the integration is an entry-by-entry integration.

We have proved

Proposition 3.8. *Suppose that $A(t)$ and $B(t)$ are continuous on (r_1, r_2), and that $t_0 \in (r_1, r_2)$. Then the solution of the system* (3.18) *with initial condition $x(t_0) = x_0$ is given by*

$$x(t) = \Omega(t)x_0 + \int_{t_0}^{t} \Omega(t)\Omega^{-1}(s)B(s)\,\mathrm{d}s, \quad t \in (r_1, r_2),$$

where $\Omega(t)$ is a fundamental matrix of the system (3.17) *satisfying $\Omega(t_0) = I$.*

In practice, Proposition 3.8 is rather useless, in the sense that any application will involve finding a suitable fundamental matrix $\Omega(t)$ and its inverse, and then performing the required integration. However, Proposition 3.8 is very useful, for it enables us to derive considerable information concerning the solution of the system (3.18) simply from the properties of $\Omega(t)$ and the behaviour of $B(t)$.

REMARK. Consider the system $x' = Ax$, where A is a constant matrix. Suppose that $\Omega(t)$ is a fundamental matrix of the system satisfying $\Omega(0) = I$. Let $s \in \mathbb{R}$. It is not difficult to see that both $\varphi_1(t) = \Omega(t)\Omega^{-1}(s)$ and $\varphi_2(t) = \Omega(t-s)$ satisfy the matrix differential equation $\Phi'(t) = A\Phi(t)$, and that $\varphi_1(s) = \varphi_2(s) = I$. By uniqueness of solution, we must have

$$\Omega(t-s) = \Omega(t)\Omega^{-1}(s). \tag{3.24}$$

Consider now the system $x' = Ax + B(t)$, where A is a constant matrix and $B(t)$ is continuous on (r_1, r_2). Suppose that $0 \in (r_1, r_2)$. Then in view of Proposition 3.8 and (3.24), the solution of this system with initial condition $x(0) = x_0$ is given by

$$x(t) = \Omega(t)x_0 + \int_0^t \Omega(t-s)B(s)\,\mathrm{d}s, \qquad t \in (r_1, r_2).$$

EXAMPLE. Recall the example at the end of Section 3.3, where it is shown that

$$\Omega(t) = \frac{1}{4}\begin{pmatrix} 3\mathrm{e}^t + \mathrm{e}^{5t} & -3\mathrm{e}^t + 3\mathrm{e}^{5t} \\ -\mathrm{e}^t + \mathrm{e}^{5t} & \mathrm{e}^t + 3\mathrm{e}^{5t} \end{pmatrix}$$

is a fundamental matrix of the two-dimensional system $x' = 2x + 3y$ and $y' = x + 4y$, and that $\Omega(0) = I$. Consider now the non-homogeneous system $x' = 2x + 3y + \mathrm{e}^{2t}$ and $y' = x + 4y + t$. Then solutions exist and are uniquely defined for $t \in (-\infty, \infty)$. Suppose that we impose the initial conditions $x(0) = x_0$ and $y(0) = y_0$. Then the solution of the non-homogeneous system with the given initial conditions is given by

$$\begin{aligned} \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \frac{1}{4}\begin{pmatrix} 3\mathrm{e}^t + \mathrm{e}^{5t} & -3\mathrm{e}^t + 3\mathrm{e}^{5t} \\ -\mathrm{e}^t + \mathrm{e}^{5t} & \mathrm{e}^t + 3\mathrm{e}^{5t} \end{pmatrix}\begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\ &\quad + \frac{1}{4}\int_0^t \begin{pmatrix} 3\mathrm{e}^{t-s} + \mathrm{e}^{5(t-s)} & -3\mathrm{e}^{t-s} + 3\mathrm{e}^{5(t-s)} \\ -\mathrm{e}^{t-s} + \mathrm{e}^{5(t-s)} & \mathrm{e}^{t-s} + 3\mathrm{e}^{5(t-s)} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{2s} \\ s \end{pmatrix}\mathrm{d}s. \end{aligned}$$

The calculations are rather unpleasant.

Problems for Chapter 3

1. We can complete the proof of Proposition 3.4 as follows. Let $\varphi_1(t),\ldots,\varphi_n(t)$ be the fundamental system of solutions of (3.3) in the proof of Proposition 3.3, and that $\psi_1(t),\ldots,\psi_n(t)$ is also a fundamental system of solutions of (3.3). To complete the proof of Proposition 3.4, we shall prove that every solution of (3.3) is a linear combination of $\psi_1(t),\ldots,\psi_n(t)$ as follows:

(i) Explain why there exist $\alpha_{11},\ldots,\alpha_{n1}\in\mathbb{R}$, not all zero, such that

$$\psi_1(t)=\alpha_{11}\varphi_1(t)+\ldots+\alpha_{n1}\varphi_n(t).$$

(ii) Without loss of generality, assume that $\alpha_{11}\neq 0$. Show that every solution of (3.3) is a linear combination of the solutions

$$\psi_1(t),\varphi_2(t),\ldots,\varphi_n(t).$$

(iii) Suppose now that $2\leqslant j\leqslant n$, and that every solution of (3.3) is a linear combination of

$$\psi_1(t),\ldots,\psi_{j-1}(t),\varphi_j(t),\ldots,\varphi_n(t).$$

Explain why there exist $\alpha_{1j},\ldots,\alpha_{nj}\in\mathbb{R}$, not all zero, such that

$$\psi_j(t)=\sum_{k=1}^{j-1}\alpha_{kj}\psi_k(t)+\sum_{k=j}^{n}\alpha_{kj}\varphi_k(t).$$

Explain also why not all $\alpha_{jj},\ldots,\alpha_{nj}$ are zero.

(iv) Without loss of generality, assume that $\alpha_{jj}\neq 0$. Show that every solution of (3.3) is a linear combination of the solutions

$$\psi_1(t),\ldots,\psi_j(t),\varphi_{j+1}(t),\ldots,\varphi_n(t).$$

(v) Complete the proof of Proposition 3.4.

2. Consider the first order system

$$\begin{aligned} x'&=3x-y+1,\\ y'&=4x-y+t.\end{aligned}$$

(i) Show that $\varphi_1(t)=(\mathrm{e}^t,2\mathrm{e}^t)$ and $\varphi_2(t)=(t\mathrm{e}^t,-\mathrm{e}^t+2t\mathrm{e}^t)$ form a fundamental system of solutions of the corresponding homogeneous system.

(ii) Find the solution $\varphi(t)$ of the given system satisfying the initial condition $\varphi(0)=(1,0)$.