

CHAPTER 4

Higher Order Linear Equations

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4.1. Homogeneous Equations

Consider the n -th order homogeneous linear differential equation

$$(4.1) \quad y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = 0,$$

where $y = y(t)$ is an unknown scalar function, and where the scalar functions $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) .

If we use the substitution

$$x_1 = y, \quad x_2 = y', \quad \dots, \quad x_n = y^{(n-1)},$$

then $x = (y, y', \dots, y^{(n-1)})$ and $x' = (y', y'', \dots, y^{(n)})$. Note that

$$\begin{pmatrix} y' \\ \vdots \\ y^{(n)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & & & \dots & 0 & 1 \\ -a_n(t) & & & \dots & -a_1(t) \end{pmatrix} \begin{pmatrix} y \\ \vdots \\ y^{(n-1)} \end{pmatrix}.$$

It follows that the equation (4.1) becomes the first order n -dimensional system

$$(4.2) \quad x' = A(t)x,$$

where

$$(4.3) \quad A(t) = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & & & \dots & 0 & 1 \\ -a_n(t) & & & \dots & -a_1(t) \end{pmatrix}.$$

Suppose further that $x_0 = (y_0, y'_0, \dots, y_0^{(n-1)})$, where $y_0, y'_0, \dots, y_0^{(n-1)}$ are given constants. Then the initial condition $x(t_0) = x_0$ of the system (4.2) is equivalent to the initial condition

$$(4.4) \quad y(t_0) = y_0, \quad y'(t_0) = y'_0, \quad \dots, \quad y^{(n-1)}(t_0) = y_0^{(n-1)}$$

of the equation (4.1).

Suppose that $y = \psi(t)$ is a solution of (4.1). Then it is not difficult to see that $x = \varphi(t) = (\psi(t), \psi'(t), \dots, \psi^{(n-1)}(t))$ is a solution of (4.2). Suppose now that $x = \varphi(t) = (\varphi_1(t), \dots, \varphi_n(t))$ is a solution of (4.2). We can show that $y = \varphi_1(t)$ is a solution of (4.1). Indeed, in view of (4.2) and (4.3), we have $y' = \varphi'_1(t) = \varphi_2(t)$, $y'' = \varphi'_2(t) = \varphi_3(t)$, $\dots$, $y^{(n-1)} = \varphi'_{n-1}(t) = \varphi_n(t)$. It follows that (4.1) and (4.2) are equivalent. We therefore have the following result.

PROPOSITION 4.1. *Suppose that the scalar functions $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) , and that $t_0 \in (r_1, r_2)$. Then given any constants $y_0, y'_0, \dots, y_0^{(n-1)}$, there exists a unique solution $y = y(t)$, $t \in (r_1, r_2)$, of the equation (4.1) such that the initial condition (4.4) is satisfied.*

We now attempt to deduce results analogous to those in Sections 3.2 and 3.3. Our task here is considerably easier in view of the equivalence of (4.1) and (4.2).

DEFINITION. Suppose that

$$(4.5) \quad \psi_1(t), \dots, \psi_n(t), \quad t \in (r_1, r_2),$$

are solutions of the n -th order differential equation (4.1), linearly independent over (r_1, r_2) . Then we say that (4.5) is a fundamental system of solutions of (4.1).

Corresponding to Proposition 3.3, we have

PROPOSITION 4.2. *For every n -th order differential equation (4.1), where the scalar functions $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) , a fundamental system of solutions exists.*

PROOF. By Proposition 3.3, a fundamental system of solutions exists for the equivalent system (4.2). Suppose that this is

$$(4.6) \quad \varphi_1(t), \dots, \varphi_n(t).$$

Furthermore, given $t_0 \in (r_1, r_2)$ and $k = 1, \dots, n$, we may assume that

$$\varphi_k(t_0) = (\underbrace{0, \dots, 0}_{k-1}, 1, \underbrace{0, \dots, 0}_{n-k}) = e_k.$$

Note now that for every $k = 1, \dots, n$,

$$\varphi_k(t) = (\psi_k(t), \psi'_k(t), \dots, \psi_k^{(n-1)}(t))$$

for some solution $\psi_k(t)$ of (4.1). It follows that the collection $\psi_1(t), \dots, \psi_n(t)$ are distinct non-trivial solutions of (4.1) (why?). It therefore remains to show that the solutions $\psi_1(t), \dots, \psi_n(t)$ are linearly independent over (r_1, r_2) . Suppose that

$$\sum_{k=1}^n \alpha_k \psi_k(t)$$

is identically zero over (r_1, r_2) . Then so are

$$\sum_{k=1}^n \alpha_k \psi'_k(t), \quad \dots, \quad \sum_{k=1}^n \alpha_k \psi_k^{(n-1)}(t)$$

(why?). It follows that

$$\sum_{k=1}^n \alpha_k \varphi_k(t)$$

is identically zero over (r_1, r_2) . Since (4.6) is a fundamental system of solutions of (4.2), we must have $\alpha_1 = \dots = \alpha_n = 0$. $\circ$

PROPOSITION 4.3. *Every solution of an n -th order differential equation (4.1), where the scalar functions $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) , is a linear combination of members of a fundamental system of solutions of (4.1).*

PROOF. Suppose that (4.5) is a fundamental system of solutions of (4.1), and that $y(t)$, $t \in (r_1, r_2)$, is a solution of (4.1) satisfying the initial condition (4.4). Consider the system of equations

$$(4.7) \quad y_0^{(j)} = y^{(j)}(t_0) = \sum_{k=1}^n \alpha_k \psi_k^{(j)}(t_0), \quad j = 0, 1, \dots, n-1,$$

in the variables $\alpha_1, \dots, \alpha_n$. Let $Y_0 = (y_0, \dots, y_0^{(n-1)})$ and $\alpha = (\alpha_1, \dots, \alpha_n)$. Then the system (4.7) can be written in the form $Y_0 = \mathcal{W}(t_0)\alpha$, where for $t \in (r_1, r_2)$,

$$(4.8) \quad \mathcal{W}(t) = \begin{pmatrix} \psi_1(t) & \dots & \psi_n(t) \\ \psi_1'(t) & \dots & \psi_n'(t) \\ \vdots & & \vdots \\ \psi_1^{(n-1)}(t) & \dots & \psi_n^{(n-1)}(t) \end{pmatrix}.$$

Note that the columns of the matrix $\mathcal{W}(t)$ form a linearly independent set of solutions of the equivalent system (4.2) in (r_1, r_2) , so that $\mathcal{W}^{-1}(t)$ exists for every $t \in (r_1, r_2)$ (why?). It follows that $\alpha = \mathcal{W}^{-1}(t_0)Y_0$. Note now that

$$y(t) = \sum_{k=1}^n \alpha_k \psi_k(t)$$

as required. $\bigcirc$

Suppose that $\psi_1(t), \dots, \psi_n(t)$, $t \in (r_1, r_2)$, are solutions of the differential equation (4.1), where the scalar functions $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) . Whether these solutions will form a fundamental system of solutions of (4.1) depends on whether they are linearly independent over (r_1, r_2) .

The determinant

$$W(t) = \det \begin{pmatrix} \psi_1(t) & \dots & \psi_n(t) \\ \psi_1'(t) & \dots & \psi_n'(t) \\ \vdots & & \vdots \\ \psi_1^{(n-1)}(t) & \dots & \psi_n^{(n-1)}(t) \end{pmatrix}$$

is called the Wronskian of the solutions $\psi_1(t), \dots, \psi_n(t)$. Also, if $\psi_1(t), \dots, \psi_n(t)$ is a fundamental system of solutions of (4.1), then the matrix (4.8) is called a fundamental matrix.

Note now that the columns of (4.8) form n solutions of the system (4.2). On the other hand, it is clear from (4.3) that $\text{tr } A(t) = -a_1(t)$. It now follows from Proposition 3.5 that

PROPOSITION 4.4. *Suppose that the scalar functions $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) . Suppose further that $\psi_1(t), \dots, \psi_n(t)$ are solutions of (4.1), and that $t_0 \in (r_1, r_2)$. Then the Wronskian of $\psi_1(t), \dots, \psi_n(t)$ is given by*

$$W(t) = W(t_0) \exp \left(- \int_{t_0}^t a_1(s) \, ds \right), \quad t \in (r_1, r_2).$$

Corresponding to Proposition 3F, we have

PROPOSITION 4.5. *Suppose that the scalar functions $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) . Suppose further that $\psi_1(t), \dots, \psi_n(t)$ are solutions of (4.1). Then $\psi_1(t), \dots, \psi_n(t)$ is a fundamental system of solutions of (4.1) if and only if the Wronskian $W(t) \neq 0$ for every $t \in (r_1, r_2)$.*

EXAMPLES. (1) Consider the second order differential equation $y'' + a(t)y = 0$, where the scalar function $a(t)$ is continuous on (r_1, r_2) . Then for any two solutions, the Wronskian $W(t)$ is constant on (r_1, r_2) , since it is clear from Proposition 4.4 that $W(t) = W(t_0)$ on (r_1, r_2) . This constant is non-zero if and only if the two solutions are linearly independent on (r_1, r_2) .

(2) Consider the third order equation

$$y''' + \frac{1}{t}y'' - \frac{2}{t^2}y' + \frac{2}{t^3}y = 0, \quad t > 0.$$

Then the Wronskian of any three solutions satisfies

$$W(t) = W(t_0) \exp \left(- \int_{t_0}^t \frac{1}{s} \, ds \right) = W(t_0) \frac{t_0}{t},$$

where $t, t_0 > 0$. A fundamental system of solutions is given by $\psi_1(t) = t^2$, $\psi_2(t) = t$ and $\psi_3(t) = 1/t$. Then

$$W(t) = \det \begin{pmatrix} t^2 & t & 1/t \\ 2t & 1 & -1/t^2 \\ 2 & 0 & 2/t^3 \end{pmatrix} = -\frac{6}{t}$$

and $W(1) = -6$.

4.2. Non-Homogeneous Equations

Consider the n -th order non-homogeneous linear differential equation

$$(4.9) \quad y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = b(t),$$

where $y = y(t)$ is an unknown scalar function, and where the scalar functions $a_1(t), \dots, a_n(t)$ and $b(t)$ are continuous on (r_1, r_2) .

Again we use the substitution $x_1 = y$, $x_2 = y'$, $\dots$, $x_n = y^{(n-1)}$. Then (4.9) can be rewritten as an n -dimensional non-homogeneous system

$$(4.10) \quad x' = A(t)x + B(t),$$

where the matrix $A(t)$ is given by (4.3), and where $B(t) = (0, \dots, 0, b(t))$.

If $x_0 = (y_0, y'_0, \dots, y_0^{(n-1)})$, where $y_0, y'_0, \dots, y_0^{(n-1)}$ are given constants, then the initial condition $x(t_0) = x_0$ of (4.10) is equivalent to the initial condition (4.4) of (4.9). We may therefore conclude that solutions of (4.9) satisfying the initial conditions (4.4) exist and are unique on (r_1, r_2) .

Suppose that $\psi_1(t), \dots, \psi_n(t)$, $t \in (r_1, r_2)$, is a fundamental system of solutions of the homogeneous equation (4.1). Consider the function

$$(4.11) \quad \psi(t) = \sum_{j=1}^n c_j(t)\psi_j(t),$$

where we shall attempt to choose $c_1(t), \dots, c_n(t)$ so that (4.11) is a solution of (4.9) in (r_1, r_2) with initial condition (4.4). Note that the matrix $\mathcal{W}(t)$ given by (4.8) is a fundamental matrix of the equivalent system (4.2). Our task is therefore simpler than choosing $c(t) = (c_1(t), \dots, c_n(t))$ so that $\Psi(t) = \mathcal{W}(t)c(t)$ is a solution of (4.10), and so

$$\mathcal{W}'(t)c(t) + \mathcal{W}(t)c'(t) = A(t)\mathcal{W}(t)c(t) + B(t).$$

Since $\mathcal{W}'(t) = A(t)\mathcal{W}(t)$, we need

$$(4.12) \quad \mathcal{W}(t)c'(t) = B(t).$$

To determine $c(t) = (c_1(t), \dots, c_n(t))$, note that $W(t) = \det \mathcal{W}(t) \neq 0$ for every $t \in (r_1, r_2)$. Noting that $B(t) = (0, \dots, 0, b(t))$, we have, applying Cramer's rule on (4.12), that

$$c'_j(t) = \frac{b(t)W_j(t)}{W(t)}, \quad j = 1, \dots, n,$$

where, for every $j = 1, \dots, n$, $W_j(t) = \det \mathcal{W}_j(t)$ and the matrix $\mathcal{W}_j(t)$ is obtained from the matrix $\mathcal{W}(t)$ by replacing the j -th column by $(0, \dots, 0, 1)$. It follows that

$$(4.13) \quad c_j(t) = c_j(t_0) + \int_{t_0}^t \frac{b(s)W_j(s)}{W(s)} ds, \quad j = 1, \dots, n.$$

Finally the initial conditions (4.4) give $\mathcal{W}(t_0)c(t_0) = Y_0 = (y_0, y'_0, \dots, y_0^{(n-1)})$, so that

$$c(t_0) = (c_1(t_0), \dots, c_n(t_0)) = \mathcal{W}^{-1}(t_0)Y_0.$$

Hence

$$\psi(t) = \sum_{j=1}^n c_j(t_0)\psi_j(t) + \sum_{j=1}^n \psi_j(t) \int_{t_0}^t \frac{b(s)W_j(s)}{W(s)} ds.$$

Note that

$$\psi_0(t) = \sum_{j=1}^n c_j(t_0)\psi_j(t)$$

is the solution of the homogeneous equation (4.1) with given initial condition (4.4).

We have proved

PROPOSITION 4.6. *Suppose that the scalar functions $a_1(t), \dots, a_n(t)$ and $b(t)$ are continuous on (r_1, r_2) , and that $\psi_1(t), \dots, \psi_n(t)$, $t \in (r_1, r_2)$, is a fundamental system of solutions of (4.1). Suppose further that $t_0 \in (r_1, r_2)$. Then the solution of the equation (4.9) with initial condition (4.4) is given by*

$$y(t) = \psi_0(t) + \sum_{j=1}^n \psi_j(t) \int_{t_0}^t \frac{b(s)W_j(s)}{W(s)} ds,$$

where $\psi_0(t)$ is the solution of the homogeneous equation (4.1) with given initial condition (4.4), and where, for every $j = 1, \dots, n$, $W_j(t) = \det \mathcal{W}_j(t)$ and the matrix $\mathcal{W}_j(t)$ is obtained from the matrix $\mathcal{W}(t)$ by replacing the j -th column by $(0, \dots, 0, 1)$.

EXAMPLES. (1) Consider the second order equation

$$(4.14) \quad y'' + a_1(t)y' + a_2(t)y = b(t),$$

where the scalar functions $a_1(t), a_2(t)$ and $b(t)$ are continuous on (r_1, r_2) . Suppose that $\psi_1(t)$ and $\psi_2(t)$ are linearly independent solutions of the homogeneous equation. Then

$$\mathcal{W}(t) = \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ \psi_1'(t) & \psi_2'(t) \end{pmatrix}.$$

Hence $W_1(t) = -\psi_2(t)$ and $W_2(t) = \psi_1(t)$. Suppose now that $t_0 \in (r_1, r_2)$. Then the solution of (4.14) with initial conditions $y(t_0) = y_0$ and $y'(t_0) = y_0'$ is given by

$$y(t) = c_1\psi_1(t) + c_2\psi_2(t) - \psi_1(t) \int_{t_0}^t \frac{b(s)\psi_2(s)}{W(s)} ds + \psi_2(t) \int_{t_0}^t \frac{b(s)\psi_1(s)}{W(s)} ds,$$

where the constants c_1 and c_2 are chosen to satisfy the initial conditions.

(2) We continue with the example at the end of Section 4.1, where $\psi_1(t) = t^2$, $\psi_2(t) = t$ and $\psi_3(t) = 1/t$ is a fundamental system of solutions of the homogeneous equation

$$y''' + \frac{1}{t}y'' - \frac{2}{t^2}y' + \frac{2}{t^3}y = 0, \quad t > 0.$$

Consider the non-homogeneous equation

$$y''' + \frac{1}{t}y'' - \frac{2}{t^2}y' + \frac{2}{t^3}y = t, \quad t > 0,$$

with initial conditions $y(1) = 1$, $y'(1) = 2$ and $y''(1) = 3$. Then (4.12) with $t = 1$ gives

$$\begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 2 & 0 & 2 \end{pmatrix} \begin{pmatrix} c_1(1) \\ c_2(1) \\ c_3(1) \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix},$$

so that $c_1(1) = 4/3$, $c_2(1) = -1/2$ and $c_3(1) = 1/6$. On the other hand,

$$W_1(t) = \det \begin{pmatrix} 0 & t & 1/t \\ 0 & 1 & -1/t^2 \\ 1 & 0 & 2/t^3 \end{pmatrix} = -\frac{2}{t},$$

$$W_2(t) = \det \begin{pmatrix} t^2 & 0 & 1/t \\ 2t & 0 & -1/t^2 \\ 2 & 1 & 2/t^3 \end{pmatrix} = 3,$$

$$W_3(t) = \det \begin{pmatrix} t^2 & t & 0 \\ 2t & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix} = -t^2.$$

Recall also that $W(t) = -6/t$. It follows from (4.13) that

$$\begin{aligned} c_1(t) &= \frac{4}{3} + \frac{1}{3} \int_1^t s \, ds = \frac{1}{6}t^2 + \frac{7}{6}, \\ c_2(t) &= -\frac{1}{2} - \frac{1}{2} \int_1^t s^2 \, ds = -\frac{1}{6}t^3 - \frac{1}{3}, \\ c_3(t) &= \frac{1}{6} + \frac{1}{6} \int_1^t s^4 \, ds = \frac{1}{30}t^5 + \frac{2}{15}. \end{aligned}$$

It now follows from (4.11) that the required solution is

$$y(t) = \left(\frac{1}{6}t^2 + \frac{7}{6}\right)t^2 + \left(-\frac{1}{6}t^3 - \frac{1}{3}\right)t + \left(\frac{1}{30}t^5 + \frac{2}{15}\right)\frac{1}{t} = \frac{1}{30}t^4 + \frac{7}{6}t^2 - \frac{1}{3}t + \frac{2}{15t}.$$

Problems for Chapter 4

1. Consider the differential equation $y'' + a(t)y = 0$, where the function $a(t)$ is continuous and real valued on (r_1, r_2) . Let $t_0 \in (r_1, r_2)$. Suppose that ψ is a non-trivial solution of the equation satisfying $\psi(t_0) = 0$. Show that $\psi'(t_0) \neq 0$.

2. Consider the differential equation $y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = 0$, where $a_1(t), \dots, a_n(t)$ are continuous and real valued on (r_1, r_2) . Let $t_0 \in (r_1, r_2)$.

(i) Show that if ψ is a solution of the equation, then so are $\Re \psi$ and $\Im \psi$.

(ii) Suppose that ψ is a solution of the equation. Suppose also that $\psi(t_0), \psi'(t_0), \dots, \psi^{(n-1)}(t_0)$ are all real. Prove that ψ is real valued on (r_1, r_2) .

3. Consider the differential equation $y'' + a_1(t)y' + a_2(t)y = 0$, where the functions $a_1(t)$ and $a_2(t)$ are continuous on an interval (r_1, r_2) . Suppose that $\psi_1(t)$ and $\psi_2(t)$ form a fundamental system of solutions. Show that

$$a_1(t) = -\frac{\det \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ \psi_1'(t) & \psi_2'(t) \end{pmatrix}}{\det \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ \psi_1'(t) & \psi_2'(t) \end{pmatrix}} \quad \text{and} \quad a_2(t) = \frac{\det \begin{pmatrix} \psi_1'(t) & \psi_2'(t) \\ \psi_1''(t) & \psi_2''(t) \end{pmatrix}}{\det \begin{pmatrix} \psi_1(t) & \psi_2(t) \\ \psi_1'(t) & \psi_2'(t) \end{pmatrix}}.$$

4. Suppose that $u_1(t), \dots, u_n(t)$ are real valued scalar functions continuous on the closed interval $[r_1, r_2]$. Suppose further that $M = (m_{ij})$ is an $n \times n$ matrix, where for every $i, j = 1, \dots, n$,

$$m_{ij} = \int_{r_1}^{r_2} u_i(t)u_j(t) dt.$$

(i) Show that if $u_1(t), \dots, u_n(t)$ are linearly dependent over (r_1, r_2) , then $\det M = 0$.

(ii) Suppose that $\det M = 0$. Then the rows of M are linearly dependent. Let $b_1, \dots, b_n \in \mathbb{R}$, not all zero, satisfy $b_1 m_{1j} + \dots + b_n m_{nj} = 0$ for every $j = 1, \dots, n$, and let

$$A(t) = b_1 u_1(t) + \dots + b_n u_n(t)$$

for every $t \in [r_1, r_2]$. Show that

$$\int_{r_1}^{r_2} A^2(t) dt = 0.$$

Deduce that $u_1(t), \dots, u_n(t)$ are linearly dependent over (r_1, r_2) .

5. Consider the differential equation

$$y'' + \frac{1}{t}y' - \frac{1}{t^2}y = 0, \quad t > 0.$$

(i) Show that there is a solution of the form t^k , where k is a constant.

(ii) Find two linearly independent solutions of the equation, and show that they are linearly independent.

(iii) Find the solution $y(t)$ of the equation satisfying $y(1) = 1$ and $y'(1) = 0$.

(iv) Find the solution $y(t)$ of the equation satisfying $y(1) = 0$ and $y'(1) = 1$.

6. Consider the n -th order linear differentiation equation

$$y^{(n)} + a_1(t)y^{(n-1)} + \dots + a_n(t)y = 0,$$

where $a_1(t), \dots, a_n(t)$ are continuous on (r_1, r_2) . Suppose that $y_1(t)$ is a solution of the equation, and that $t_0 \in (r_1, r_2)$. Show that the substitution

$$y = y_1(t) \int_{t_0}^t u(s) ds$$

will result in an $(n-1)$ -th order linear differentiation equation in the variable $u = u(t)$. Use this method to find a second solution to each of the following equations, linearly independent of the given $y_1(t)$:

(i) $y'' + 4ty' + (4t^2 + 2)y = 0$ and $y_1(t) = e^{-t^2}$

(ii) $y'' - (2 \sec^2 t)y = 0$ and $y_1(t) = \tan t$

(iii) $ty'' - (t+1)y' - 2(t-1)y = 0$ and $y_1(t) = e^{2t}$

7. Suppose that $a(t)$ is continuous on (r_1, r_2) , and that the scalar function $y_1(t)$ is a non-trivial solution of the differential equation $y'' + a(t)y = 0$. Show that if $t_0 \in (r_1, r_2)$, then

$$y_2(t) = y_1(t) \int_{t_0}^t (y_1(s))^{-2} ds$$

is another solution, linearly independent from $y_1(t)$ over (r_1, r_2) .

8. Consider the differential equation $y'' + a_1(t)y' + a_2(t)y = 0$, where the functions $a_1(t)$ and $a_2(t)$ are continuous on an interval (r_1, r_2) , and $a_1(t)$ has continuous derivative on (r_1, r_2) .

- (i) Suppose that ψ is a solution of the equation, and that $\psi = u\varphi$, where $u = u(t)$ and $\varphi = \varphi(t)$ are both twice-differentiable on (r_1, r_2) . Find a differential equation in $u = u(t)$ which will make φ the solution of a differential equation in which the term involving the first derivative is absent.
- (ii) Solve this differential equation for u .
- (iii) Show that φ is a solution of the differential equation $x'' + \alpha(t)x = 0$, where

$$\alpha(t) = a_2(t) - \frac{1}{4}a_1^2(t) - \frac{1}{2}a_1'(t).$$

9. Consider the differential equation $y'' + a_1(t)y' + a_2(t)y = 0$, where the functions $a_1(t)$ and $a_2(t)$ are continuous on an interval (r_1, r_2) . Show that if $t_0 \in (r_1, r_2)$, then the function

$$\psi(t) = \exp\left(\int_{t_0}^t p(s) ds\right)$$

is a solution of the equation if and only if $p(t)$ is a solution of the *Riccati equation*

$$x' = -x^2 - a_1(t)x - a_2(t).$$

10. Suppose that $a_1(t)$ and $a_2(t)$ are continuous on (r_1, r_2) . Suppose also that ψ_1 is a real valued non-trivial solution of the differential equation $y'' + a_1(t)y = 0$, and ψ_2 is a real valued non-trivial solution of the differential equation $y'' + a_2(t)y = 0$. Suppose further that $a_2(t) > a_1(t)$ for every $t \in (r_1, r_2)$. Follow the steps indicated below to show that if $t_1 < t_2$ are successive zeros of ψ_1 on (r_1, r_2) , then $\psi_2(t) = 0$ for some $t \in (t_1, t_2)$. Suppose on the contrary that $\psi_2(t) \neq 0$ in (t_1, t_2) . In view of the Intermediate value theorem, we may assume, without loss of generality, that $\psi_2(t) > 0$ in (t_1, t_2) . Also, in view of the Intermediate value theorem, either $\psi_1(t) > 0$ for all $t \in (t_1, t_2)$ or $\psi_1(t) < 0$ for all $t \in (t_1, t_2)$. Consider the case $\psi_1(t) > 0$ for all $t \in (t_1, t_2)$ (the case $\psi_1(t) < 0$ for all $t \in (t_1, t_2)$ is essentially similar).

- (i) Use your knowledge in analysis to show that $\psi_1(t_1) > 0$ and $\psi_1(t_2) < 0$.
- (ii) Show that $(\psi_2(t)\psi_1'(t) - \psi_1(t)\psi_2'(t))' = (a_2(t) - a_1(t))\psi_1(t)\psi_2(t)$.
- (iii) Show that $\psi_2(t_2)\psi_1'(t_2) - \psi_2(t_1)\psi_1'(t_1) > 0$.
- (iv) Point out the contradiction.

11. Use the idea in Question 10 to show that any real valued solution of the differential equation $y'' + ty = 0$ has infinitely many zeros on $(0, \infty)$.

[Hint: Let $a_1(t) = 1$, $a_2(t) = t$ and $\psi_1(t) = \cos t$.]

12. Consider the differential equation

$$y'' + \frac{4}{t}y' + \frac{2}{t^2}y = \frac{\sin t}{t}, \quad t > 0.$$

Show that $y_1(t) = 1/t$ and $y_2(t) = 1/t^2$ form a fundamental system of solutions of the corresponding homogeneous equation. Find also the solution $y(t)$ of the given equation satisfying $y(1) = 1$ and $y'(1) = 0$.

13. Consider the differential equation

$$y'' - \frac{2}{t}y' + y = t^2, \quad t > 0.$$

Show that $y_1(t) = \sin t - t \cos t$ and $y_2(t) = \cos t + t \sin t$ form a fundamental system of solutions of the corresponding homogeneous equation. Find also the solution $y(t)$ of the given equation satisfying $y(\pi/2) = 0$ and $y'(\pi/2) = 1$.

14. Show that the function $\psi(t)$, defined by (4.11), satisfies

$$\psi^{(k)}(t_0) = \sum_{j=1}^n c_j(t_0) \psi_j^{(k)}(t_0), \quad k = 0, 1, \dots, n.$$

15. Consider the differential equation

$$t^2 y'' + 4ty' + (2 + t^2)y = t^2, \quad t > 0.$$

(i) Show that a fundamental system of solutions of the corresponding homogeneous equation is given by

$$\psi_1(t) = \frac{\varphi_1(t)}{t^2} \quad \text{and} \quad \psi_2(t) = \frac{\varphi_2(t)}{t^2},$$

where $\varphi_1(t)$ and $\varphi_2(t)$ form a fundamental system of solutions of the equation $x'' + x = 0$.

(ii) Find all solutions of the original equation for $t > 0$.