CHAPTER 5

Linear Equations with Constant Coefficients

5.1. Homogeneous Equations

Consider the n-th order homogeneous linear differential equation

$$z^{(n)} + a_1 z^{(n-1)} + \ldots + a_n z = 0, \tag{5.1}$$

where $z = z(t)$ is an unknown real or complex valued scalar function, and where $a_1, \ldots, a_n$ are real or complex constants.

The importance of equations of the type (5.1) is two-fold. Firstly, the determination of a fundamental system of solutions merely involves finding the roots of an associated polynomial. Secondly, many physical phenomena can be described in terms of such equations, after perhaps a *linearization* process.

Since the coefficients $a_1, \ldots, a_n$ are constants, the following two results follow immediately from Propositions 4.2, 4.3 and 4.4.

PROPOSITION 5.1. *For every n-th order differential equation* (5.1)*, a fundamental system of solutions exists, and every solution is a linear combination of members of such a fundamental system.*

PROPOSITION 5.2. *Suppose that $\psi_1(t), \ldots, \psi_n(t)$ are solutions of* (5.1)*. Then the Wronskian of $\psi_1(t), \ldots, \psi_n(t)$ is given by*

$$W(t) = W(0)\mathrm{e}^{-a_1 t}, \quad t \in (-\infty, \infty).$$

We introduce some notation. We define the differential operator D recursively by $Dz = z'$ and $D^k z = D(D^{k-1} z)$ for $k \geqslant 2$. Furthermore, if

$$L(p) = p^n + a_1 p^{n-1} + \ldots + a_n \tag{5.2}$$

is a polynomial, then we define the operator $L(D) = D^n + a_1 D^{n-1} + \ldots + a_n$ by writing

$$L(D)z = D^n z + a_1 D^{n-1} z + \ldots + a_n z = z^{(n)} + a_1 z^{(n-1)} + \ldots + a_n z,$$

so that equation (5.1) can be denoted by

$$L(D)z = 0.$$

It is easy to see that the following properties hold:

(i) If a and b are any constants and if z_1 and z_2 are any sufficiently differentiable scalar functions, then $L(D)(az_1 + bz_2) = aL(D)z_1 + bL(D)z_2$.

(ii) If $M(p)$ is another polynomial and if z is a sufficiently differentiable scalar function, then $(L + M)(D)z = L(D)z + M(D)z$.

We call the polynomial (5.2) the characteristic polynomial of the equation (5.1).

PROPOSITION 5.3. *Suppose that $L(p)$ is any arbitrary polynomial. Then*

$$L(D)\mathrm{e}^{\lambda t} = L(\lambda)\mathrm{e}^{\lambda t}.$$

PROOF. Note that

$$\begin{aligned} L(D)\mathrm{e}^{\lambda t} &= D^n\mathrm{e}^{\lambda t} + a_1 D^{n-1}\mathrm{e}^{\lambda t} + \ldots + a_n\mathrm{e}^{\lambda t} = \frac{\mathrm{d}^n}{\mathrm{d}t^n}\mathrm{e}^{\lambda t} + a_1\frac{\mathrm{d}^{n-1}}{\mathrm{d}t^{n-1}}\mathrm{e}^{\lambda t} + \ldots + a_n\mathrm{e}^{\lambda t} \\ &= \lambda^n\mathrm{e}^{\lambda t} + a_1\lambda^{n-1}\mathrm{e}^{\lambda t} + \ldots + a_n\mathrm{e}^{\lambda t} = (\lambda^n + a_1\lambda^{n-1} + \ldots + a_n)\mathrm{e}^{\lambda t} = L(\lambda)\mathrm{e}^{\lambda t} \end{aligned}$$

as required. ○

REMARK. Note that we have used the result

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{e}^{\lambda t} = \lambda\mathrm{e}^{\lambda t}$$

in the proof. Make sure you can justify this for any complex number λ.

An immediate consequence of Proposition 5.3 is the following important result.

PROPOSITION 5.4. *The function $\psi(t) = \mathrm{e}^{\lambda t}$ is a solution of the differential equation* (5.1) *if and only if λ is a root of the characteristic polynomial* (5.2).

In the next two sections, we shall justify our claim that the determination of a fundamental system of solutions merely involves finding the roots of an associated polynomial.

5.2. Distinct Roots

PROPOSITION 5.5. *Suppose that the characteristic polynomial* (5.2) *has distinct roots $\lambda_1, \ldots, \lambda_n$. Then the functions*

$$\mathrm{e}^{\lambda_1 t}, \ldots, \mathrm{e}^{\lambda_n t} \tag{5.3}$$

form a fundamental system of solutions of the differential equation (5.1).

PROOF. In view of Proposition 5.4, the functions (5.3) are all solutions of (5.1). To show that they form a fundamental system of solutions of (5.1), it remains to show, in view of Proposition 4.5, that the Wronskian $W(t)$ does not vanish for any $t \in (-\infty, \infty)$. In view of Proposition 5.2, it suffices to show that $W(0) \neq 0$. Now

$$W(0) = \det\begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \lambda_1^2 & \ldots & \lambda_n^2 \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix} = \prod_{1\leqslant i<j\leqslant n}(\lambda_j - \lambda_i) \neq 0$$

since $\lambda_1, \ldots, \lambda_n$ are distinct. ○

REMARK. The determinant

$$\det\begin{pmatrix} 1 & \ldots & 1 \\ \lambda_1 & \ldots & \lambda_n \\ \lambda_1^2 & \ldots & \lambda_n^2 \\ \vdots & & \vdots \\ \lambda_1^{n-1} & \ldots & \lambda_n^{n-1} \end{pmatrix} \tag{5.4}$$

is called a Vandermonde determinant, and can be calculated by elementary row operations as follows. Subtracting λ_1 times the $(n-1)$-th row from the n-th row, then subtracting λ_1 times the $(n-2)$-th row from the $(n-1)$-th row, and so on, and finally subtracting λ_1 times the first row from the second row, we see that (5.4) is reduced to

$$\det\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \lambda_2-\lambda_1 & \dots & \lambda_n-\lambda_1 \\ 0 & \lambda_2^2-\lambda_1\lambda_2 & \dots & \lambda_n^2-\lambda_1\lambda_n \\ \vdots & \vdots & & \vdots \\ 0 & \lambda_2^{n-1}-\lambda_1\lambda_2^{n-2} & \dots & \lambda_n^{n-1}-\lambda_1\lambda_n^{n-2} \end{pmatrix} \tag{5.5}$$

$$= \det\begin{pmatrix} 1 & 1 & \dots & 1 \\ 0 & \lambda_2-\lambda_1 & \dots & \lambda_n-\lambda_1 \\ 0 & \lambda_2(\lambda_2-\lambda_1) & \dots & \lambda_n(\lambda_n-\lambda_1) \\ \vdots & \vdots & & \vdots \\ 0 & \lambda_2^{n-2}(\lambda_2-\lambda_1) & \dots & \lambda_n^{n-2}(\lambda_n-\lambda_1) \end{pmatrix}.$$

Expanding by the first column, we see that (5.5) is reduced to

$$\det\begin{pmatrix} \lambda_2-\lambda_1 & \dots & \lambda_n-\lambda_1 \\ \lambda_2(\lambda_2-\lambda_1) & \dots & \lambda_n(\lambda_n-\lambda_1) \\ \vdots & & \vdots \\ \lambda_2^{n-2}(\lambda_2-\lambda_1) & \dots & \lambda_n^{n-2}(\lambda_n-\lambda_1) \end{pmatrix}.$$

This is equal to

$$(\lambda_2-\lambda_1)\dots(\lambda_n-\lambda_1)\det\begin{pmatrix} 1 & \dots & 1 \\ \lambda_2 & \dots & \lambda_n \\ \vdots & & \vdots \\ \lambda_2^{n-2} & \dots & \lambda_n^{n-2} \end{pmatrix}. \tag{5.6}$$

The $(n-1)\times(n-1)$ determinant in (5.6) is also a Vandermonde determinant, so we can repeat the argument to show that it is equal to

$$(\lambda_3-\lambda_2)\dots(\lambda_n-\lambda_2)\det\begin{pmatrix} 1 & \dots & 1 \\ \lambda_3 & \dots & \lambda_n \\ \vdots & & \vdots \\ \lambda_3^{n-3} & \dots & \lambda_n^{n-3} \end{pmatrix}.$$

And so on.

Proposition 5.6. *Suppose that the characteristic polynomial* (5.2) *has real coefficients. Suppose further that it has distinct real roots* λ_j, $j=1,\dots,k$, *and distinct pairs of non-real roots* $\lambda_j^+ = u_j + \mathrm{i}v_j$ *and* $\lambda_j^- = u_j - \mathrm{i}v_j$, $j=k+1,\dots,s$. *Then the real valued functions*

$$\mathrm{e}^{\lambda_1 t},\dots,\mathrm{e}^{\lambda_k t},$$

together with the real valued functions

$$\mathrm{e}^{u_{k+1}t}\cos v_{k+1}t,\dots,\mathrm{e}^{u_s t}\cos v_s t \quad \textit{and} \quad \mathrm{e}^{u_{k+1}t}\sin v_{k+1}t,\dots,\mathrm{e}^{u_s t}\sin v_s t, \tag{5.7}$$

form a fundamental system of solutions of the differential equation (5.1).

Remark. Note that $n = k + 2(s-k)$.

Proof of Proposition 5.6. By Proposition 5.5, the functions

$$\mathrm{e}^{\lambda_1 t},\dots,\mathrm{e}^{\lambda_k t},$$

together with the functions

$$\mathrm{e}^{\lambda_{k+1}^+ t},\dots,\mathrm{e}^{\lambda_s^+ t} \quad \text{and} \quad \mathrm{e}^{\lambda_{k+1}^- t},\dots,\mathrm{e}^{\lambda_s^- t}, \tag{5.8}$$

form a fundamental system of solutions of (5.1). For every $j=k+1,\dots,s$, note now that

$$\begin{pmatrix} \mathrm{e}^{\lambda_j^+ t} \\ \mathrm{e}^{\lambda_j^- t} \end{pmatrix} = \begin{pmatrix} 1 & \mathrm{i} \\ 1 & -\mathrm{i} \end{pmatrix}\begin{pmatrix} \mathrm{e}^{u_j t}\cos v_j t \\ \mathrm{e}^{u_j t}\sin v_j t \end{pmatrix}$$

and

$$\begin{pmatrix} \mathrm{e}^{u_j t}\cos v_j t \\ \mathrm{e}^{u_j t}\sin v_j t \end{pmatrix} = \begin{pmatrix} 1/2 & 1/2 \\ -\mathrm{i}/2 & \mathrm{i}/2 \end{pmatrix} \begin{pmatrix} \mathrm{e}^{\lambda_j^+ t} \\ \mathrm{e}^{\lambda_j^- t} \end{pmatrix}.$$

It follows that linear independence is preserved if we replace (5.8) by (5.7). ○

5.3. Repeated Roots

PROPOSITION 5.7. *Suppose that the characteristic polynomial* (5.2) *has distinct roots* $\lambda_1, \ldots, \lambda_k$*, with multiplicities* $\mu_1, \ldots, \mu_k$ *respectively. Then the functions*

$$\mathrm{e}^{\lambda_1 t}, t\mathrm{e}^{\lambda_1 t}, \ldots, t^{\mu_1 - 1}\mathrm{e}^{\lambda_1 t}, \quad \ldots, \quad \mathrm{e}^{\lambda_k t}, t\mathrm{e}^{\lambda_k t}, \ldots, t^{\mu_k - 1}\mathrm{e}^{\lambda_k t} \tag{5.9}$$

form a fundamental system of solutions of the differential equation (5.1)*.*

REMARK. Note that $n = \mu_1 + \ldots + \mu_k$.

PROOF OF PROPOSITION 5.7. The proof is divided into two parts. We first show that every function in (5.9) is a solution of (5.1). We then show that the functions in (5.9) form a fundamental system of solutions by investigating the Wronskian.

(i) For every $j = 1, \ldots, k$, we need to show that

$$L(D)t^r \mathrm{e}^{\lambda_j t} = 0, \quad r = 0, 1, \ldots, \mu_j - 1. \tag{5.10}$$

If $\mu_j = 1$, then clearly $\mathrm{e}^{\lambda_j t}$ is a solution of (5.1). Suppose now that $\mu_j > 1$. Then since $L(p) = (p - \lambda_j)^{\mu_j} M(p)$ for some polynomial $M(p)$, it follows from applying Leibnitz's rule and evaluating at $p = \lambda_j$ that

$$\frac{\mathrm{d}^r}{\mathrm{d}p^r} L(\lambda_j) = 0, \quad r = 0, 1, \ldots, \mu_j - 1. \tag{5.11}$$

Consider the function $F(p,t) = \mathrm{e}^{pt}$. This has continuous partial derivatives of all orders. It follows that for every $r, s \geqslant 0$,

$$\frac{\partial^r}{\partial p^r}\frac{\partial^s}{\partial t^s} F(p,t) = \frac{\partial^s}{\partial t^s}\frac{\partial^r}{\partial p^r} F(p,t). \tag{5.12}$$

The left hand side of (5.12) is equal to

$$\frac{\partial^r}{\partial p^r}\frac{\partial^s}{\partial t^s}\mathrm{e}^{pt} = \frac{\partial^r}{\partial p^r}(p^s \mathrm{e}^{pt}),$$

while the right hand side of (5.12) is equal to

$$\frac{\partial^s}{\partial t^s}\frac{\partial^r}{\partial p^r}\mathrm{e}^{pt} = \frac{\partial^s}{\partial t^s}(t^r \mathrm{e}^{pt}) = (t^r \mathrm{e}^{pt})^{(s)}.$$

It follows that for every $r, s \geqslant 0$,

$$\frac{\partial^r}{\partial p^r}(p^s \mathrm{e}^{pt}) = (t^r \mathrm{e}^{pt})^{(s)}. \tag{5.13}$$

Consider now the function $\varphi(p,t) = L(D)\mathrm{e}^{pt} = L(p)\mathrm{e}^{pt}$. By Leibnitz's rule, for any $r \geqslant 0$,

$$\begin{aligned} \frac{\partial^r \varphi}{\partial p^r} &= \frac{\partial^r}{\partial p^r}(L(p)\mathrm{e}^{pt}) \\ &= \left(\frac{\mathrm{d}^r}{\mathrm{d}p^r}L(p)\right)\mathrm{e}^{pt} + \binom{r}{1}\left(\frac{\mathrm{d}^{r-1}}{\mathrm{d}p^{r-1}}L(p)\right)\left(\frac{\partial}{\partial p}\mathrm{e}^{pt}\right) + \ldots + L(p)\left(\frac{\mathrm{d}^r}{\mathrm{d}p^r}\mathrm{e}^{pt}\right) \\ &= \mathrm{e}^{pt}\left(\frac{\mathrm{d}^r}{\mathrm{d}p^r}L(p) + rt\frac{\mathrm{d}^{r-1}}{\mathrm{d}p^{r-1}}L(p) + \ldots + t^r L(p)\right). \end{aligned} \tag{5.14}$$

On the other hand, in view of (5.13),

$$\begin{aligned}\frac{\partial^r\varphi}{\partial p^r} &= \frac{\partial^r}{\partial p^r}(L(p)\mathrm{e}^{pt}) = \frac{\partial^r}{\partial p^r}(p^n\mathrm{e}^{pt} + a_1p^{n-1}\mathrm{e}^{pt} + \ldots + a_n\mathrm{e}^{pt}) \\ &= \frac{\partial^r}{\partial p^r}(p^n\mathrm{e}^{pt}) + a_1\frac{\partial^r}{\partial p^r}(p^{n-1}\mathrm{e}^{pt}) + \ldots + a_n\frac{\partial^r}{\partial p^r}(\mathrm{e}^{pt}) \\ &= (t^r\mathrm{e}^{pt})^{(n)} + a_1(t^r\mathrm{e}^{pt})^{(n-1)} + \ldots + a_n(t^r\mathrm{e}^{pt}) \\ &= L(D)t^r\mathrm{e}^{pt}.\end{aligned} \tag{5.15}$$

Combining (5.14) and (5.15), we have

$$L(D)t^r\mathrm{e}^{pt} = \mathrm{e}^{pt}\left(\frac{\mathrm{d}^r}{\mathrm{d}p^r}L(p) + rt\frac{\mathrm{d}^{r-1}}{\mathrm{d}p^{r-1}}L(p) + \ldots + t^rL(p)\right). \tag{5.16}$$

In view of (5.11), it is clear that the right hand side of (5.16) vanishes with $p = \lambda_j$ if $0 \leqslant r < \mu_j$. This establishes (5.10).

(ii) Consider the Wronskian $W(t)$ of the functions (5.9). In view of Propositions 4.5 and 5.2, it suffices to show that $W(0) \neq 0$. Now

$$W(0) = \det\left(\begin{array}{c|c|c} I_1 & \ldots & I_k \end{array}\right), \tag{5.17}$$

where, for every $j = 1, \ldots, k$, the $n \times \mu_j$ matrix

$$I_j = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ \lambda_j & 1 & 0 & \ldots & 0 \\ \lambda_j^2 & \binom{2}{1}\lambda_j & 1 & \ldots & 0 \\ \lambda_j^3 & \binom{3}{1}\lambda_j^2 & \binom{3}{2}\lambda_j & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_j^{\mu_j-1} & \binom{\mu_j-1}{1}\lambda_j^{\mu_j-2} & \binom{\mu_j-1}{2}\lambda_j^{\mu_j-3} & \ldots & 1 \\ \vdots & \vdots & \vdots & & \vdots \\ \lambda_j^{n-1} & \binom{n-1}{1}\lambda_j^{n-2} & \binom{n-1}{2}\lambda_j^{n-3} & \ldots & \binom{n-1}{\mu_j-1}\lambda_j^{n-\mu_j} \end{pmatrix}.$$

Now

$$W(0) = \prod_{1\leqslant i<j\leqslant k}(\lambda_j - \lambda_i)^{\mu_i\mu_j} \neq 0$$

since $\lambda_1, \ldots, \lambda_n$ are distinct. ◯

Remark. The matrix in (5.17) is often called a generalized Vandermonde matrix. The determinant (5.17) is often called a generalized Vandermonde determinant, and can be calculated by elementary row and column operations as follows. First of all, observe that for every $j = 1, \ldots, k$, the $n \times \mu_j$ matrix

$$I_j = \begin{pmatrix} \binom{0}{0} & 0 & 0 & \ldots & 0 \\ \binom{1}{0}\lambda_j & \binom{1}{1} & 0 & \ldots & 0 \\ \binom{2}{0}\lambda_j^2 & \binom{2}{1}\lambda_j & \binom{2}{2} & \ldots & 0 \\ \binom{3}{0}\lambda_j^3 & \binom{3}{1}\lambda_j^2 & \binom{3}{2}\lambda_j & \ldots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{\mu_j-1}{0}\lambda_j^{\mu_j-1} & \binom{\mu_j-1}{1}\lambda_j^{\mu_j-2} & \binom{\mu_j-1}{2}\lambda_j^{\mu_j-3} & \ldots & \binom{\mu_j-1}{\mu_j-1} \\ \vdots & \vdots & \vdots & & \vdots \\ \binom{n-1}{0}\lambda_j^{n-1} & \binom{n-1}{1}\lambda_j^{n-2} & \binom{n-1}{2}\lambda_j^{n-3} & \ldots & \binom{n-1}{\mu_j-1}\lambda_j^{n-\mu_j} \end{pmatrix}.$$

Consider the matrix in (5.17). Subtracting λ_1 times the $(n-1)$-th row from the n-th row, then subtracting λ_1 times the $(n-2)$-th row from the $(n-1)$-th row, and so on, and finally subtracting

λ_1 times the first row from the second row, we get the matrix

$$\left(\begin{array}{c|c|c} J_1 & \dots & J_k \end{array} \right), \tag{5.18}$$

where the $n \times \mu_1$ matrix

$$J_1 = \begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & \binom{0}{0} & 0 & \dots & 0 \\ 0 & \binom{1}{0}\lambda_1 & \binom{1}{1} & \dots & 0 \\ 0 & \binom{2}{0}\lambda_1^2 & \binom{2}{1}\lambda_1 & \dots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ 0 & \binom{n-2}{0}\lambda_1^{n-2} & \binom{n-2}{1}\lambda_1^{n-3} & \dots & \binom{n-2}{\mu_1-2}\lambda_1^{n-\mu_1} \end{pmatrix},$$

and where, for $j = 2, \dots, k$, the $n \times \mu_j$ matrix

$$J_j = \begin{pmatrix} 1 & 0 & \dots & \\ \lambda_j - \lambda_1 & 1 & \dots & \\ \lambda_j^2 - \lambda_1\lambda_j & \binom{2}{1}\lambda_j - \binom{1}{1}\lambda_1 & \dots & \\ \vdots & \vdots & & \\ \lambda_j^{n-2} - \lambda_1\lambda_j^{n-3} & \binom{n-2}{1}\lambda_j^{n-3} - \binom{n-3}{1}\lambda_1\lambda_j^{n-4} & \dots & \\ \lambda_j^{n-1} - \lambda_1\lambda_j^{n-2} & \binom{n-1}{1}\lambda_j^{n-2} - \binom{n-2}{1}\lambda_1\lambda_j^{n-3} & \dots & \binom{n-1}{\mu_j-1}\lambda_j^{n-\mu_j} - \binom{n-2}{\mu_j-1}\lambda_1\lambda_j^{n-\mu_j-1} \end{pmatrix}.$$

Expanding the determinant of (5.18) by the first column, we conclude that $W(0)$ is equal to the determinant of the $(n-1) \times (n-1)$ matrix

$$\left(\begin{array}{c|c|c} J_1^* & \dots & J_k^* \end{array} \right),$$

where the $(n-1) \times (\mu_1 - 1)$ matrix

$$J_1^* = \begin{pmatrix} \binom{0}{0} & 0 & \dots & 0 \\ \binom{1}{0}\lambda_1 & \binom{1}{1} & \dots & 0 \\ \binom{2}{0}\lambda_1^2 & \binom{2}{1}\lambda_1 & \dots & 0 \\ \vdots & \vdots & & \vdots \\ \binom{n-2}{0}\lambda_1^{n-2} & \binom{n-2}{1}\lambda_1^{n-3} & \dots & \binom{n-2}{\mu_1-2}\lambda_1^{n-\mu_1} \end{pmatrix}, \tag{5.19}$$

and where, for $j = 2, \dots, k$, the $(n-1) \times (\mu_j - 1)$ matrix

$$J_j^* = \begin{pmatrix} \lambda_j - \lambda_1 & 1 & \dots & \\ \lambda_j^2 - \lambda_1\lambda_j & \binom{2}{1}\lambda_j - \binom{1}{1}\lambda_1 & \dots & \\ \vdots & \vdots & & \\ \lambda_j^{n-2} - \lambda_1\lambda_j^{n-3} & \binom{n-2}{1}\lambda_j^{n-3} - \binom{n-3}{1}\lambda_1\lambda_j^{n-4} & \dots & \\ \lambda_j^{n-1} - \lambda_1\lambda_j^{n-2} & \binom{n-1}{1}\lambda_j^{n-2} - \binom{n-2}{1}\lambda_1\lambda_j^{n-3} & \dots & \binom{n-1}{\mu_j-1}\lambda_j^{n-\mu_j} - \binom{n-2}{\mu_j-1}\lambda_1\lambda_j^{n-\mu_j-1} \end{pmatrix}.$$

We shall now perform elementary column operations on J_j^* for every $j = 2, \dots, k$. Removing a factor $(\lambda_j - \lambda_1)$ from the first column, J_j^* becomes

$$\begin{pmatrix} 1 & 1 & \dots & \\ \binom{1}{0}\lambda_j & \binom{2}{1}\lambda_j - \binom{1}{1}\lambda_1 & \dots & \\ \vdots & \vdots & & \\ \binom{n-3}{0}\lambda_j^{n-3} & \binom{n-2}{1}\lambda_j^{n-3} - \binom{n-3}{1}\lambda_1\lambda_j^{n-4} & \dots & \\ \binom{n-2}{0}\lambda_j^{n-2} & \binom{n-1}{1}\lambda_j^{n-2} - \binom{n-2}{1}\lambda_1\lambda_j^{n-3} & \dots & \binom{n-1}{\mu_j-1}\lambda_j^{n-\mu_j} - \binom{n-2}{\mu_j-1}\lambda_1\lambda_j^{n-\mu_j-1} \end{pmatrix}.$$

Subtracting now the first column here from the second column, we get

$$\begin{pmatrix} 1 & 0 & \dots & \\ \binom{1}{0}\lambda_j & \binom{1}{1}(\lambda_j-\lambda_1) & \dots & \\ \vdots & \vdots & & \\ \binom{n-3}{0}\lambda_j^{n-3} & \binom{n-3}{1}(\lambda_j-\lambda_1)\lambda_j^{n-4} & \dots & \\ \binom{n-2}{0}\lambda_j^{n-2} & \binom{n-2}{1}(\lambda_j-\lambda_1)\lambda_j^{n-3} & \dots & \binom{n-1}{\mu_j-1}\lambda_j^{n-\mu_j}-\binom{n-2}{\mu_j-1}\lambda_1\lambda_j^{n-\mu_j-1} \end{pmatrix}.$$

Removing a factor $(\lambda_j - \lambda_1)$ from the second column, this matrix now becomes

$$\begin{pmatrix} 1 & 0 & \dots & \\ \binom{1}{0}\lambda_j & 1 & \dots & \\ \vdots & \vdots & & \\ \binom{n-3}{0}\lambda_j^{n-3} & \binom{n-3}{1}\lambda_j^{n-4} & \dots & \\ \binom{n-2}{0}\lambda_j^{n-2} & \binom{n-2}{1}\lambda_j^{n-3} & \dots & \binom{n-1}{\mu_j-1}\lambda_j^{n-\mu_j}-\binom{n-2}{\mu_j-1}\lambda_1\lambda_j^{n-\mu_j-1} \end{pmatrix}.$$

We now subtract the second column here from the third column and then remove a factor $(\lambda_j - \lambda_1)$ from the third column, and so on. To summarize, for each $j = 2, \dots, k$, J_j^* becomes the $(n-1) \times \mu_j$ matrix

$$\begin{pmatrix} 1 & 0 & \dots & \\ \binom{1}{0}\lambda_j & 1 & \dots & \\ \vdots & \vdots & & \\ \binom{n-3}{0}\lambda_j^{n-3} & \binom{n-3}{1}\lambda_j^{n-4} & \dots & \\ \binom{n-2}{0}\lambda_j^{n-2} & \binom{n-2}{1}\lambda_j^{n-3} & \dots & \binom{n-2}{\mu_j-1}\lambda_j^{n-\mu_j-1} \end{pmatrix} \tag{5.20}$$

and a factor of $(\lambda_j - \lambda_1)$ has been removed from each column in the process. It now follows that $W(0)$ is

$$\prod_{j=2}^{k}(\lambda_j-\lambda_1)^{\mu_j}$$

times

$$\det\left(\begin{array}{c|c|c} I_1^* & \dots & I_k^* \end{array} \right), \tag{5.21}$$

where I_1^* is given by (5.19) and I_j^* is given by (5.20) for every $j = 2, \dots, k$. Note now that (5.21) is also a generalized Vandermonde determinant, but of dimension $(n-1) \times (n-1)$. Denote by $\mathcal{V}_{\lambda_1,\dots,\lambda_k}(\mu_1, \dots, \mu_k)$ the matrix in (5.17). Then we have shown that

$$\det \mathcal{V}_{\lambda_1,\dots,\lambda_k}(\mu_1,\dots,\mu_k) = \left(\prod_{j=2}^{k}(\lambda_j-\lambda_1)^{\mu_j}\right)\det \mathcal{V}_{\lambda_1,\dots,\lambda_k}(\mu_1-1,\mu_2,\dots,\mu_k),$$

where the matrix on the right hand side has one row and one column fewer than the matrix on the left hand side. Repeating this argument a finite number of times, we conclude that

$$\det \mathcal{V}_{\lambda_1,\dots,\lambda_k}(\mu_1,\dots,\mu_k) = \left(\prod_{j=2}^{k}(\lambda_j-\lambda_1)^{\mu_1\mu_j}\right)\det \mathcal{V}_{\lambda_1,\dots,\lambda_k}(0,\mu_2,\dots,\mu_k),$$

where the matrix on the right hand side has μ_1 rows and μ_1 columns fewer than the matrix on the left hand side, and also does not involve λ_1. Omitting reference to λ_1, we can write

$$\mathcal{V}_{\lambda_1,\dots,\lambda_k}(0,\mu_2,\dots,\mu_k) = \mathcal{V}_{\lambda_2,\dots,\lambda_k}(\mu_2,\dots,\mu_k),$$

and note that this is another generalized Vandermonde matrix. Repeating the argument thus far, we deduce that

$$\det \mathcal{V}_{\lambda_2,\dots,\lambda_k}(\mu_2,\dots,\mu_k) = \left(\prod_{j=3}^{k}(\lambda_j-\lambda_2)^{\mu_2\mu_j}\right)\det \mathcal{V}_{\lambda_3,\dots,\lambda_k}(\mu_3,\dots,\mu_k),$$

where the matrix on the right hand side has μ_2 rows and μ_2 columns fewer than the matrix on the left hand side, and also does not involve λ_2. And so on. Eventually, we obtain

$$\mathcal{V}_{\lambda_1,\ldots,\lambda_k}(\mu_1,\mu_2,\ldots,\mu_k) = \prod_{1\leqslant i<j\leqslant k}(\lambda_j-\lambda_i)^{\mu_i\mu_j}\det\mathcal{V}_{\lambda_k}(\mu_k),$$

where $\mathcal{V}_{\lambda_k}(\mu_k)$ is a lower triangular matrix with diagonal entries 1, so that $\det\mathcal{V}_{\lambda_k}(\mu_k)=1$.

Remark. A simpler, but indirect, way of showing that the functions in (5.9) are linearly independent is as follows. Suppose that

$$\sum_{j=1}^{k}\sum_{r=0}^{\mu_j-1}\alpha_{jr}t^r\mathrm{e}^{\lambda_j t}=0, \tag{5.22}$$

where α_{jr} is a constant for every $j=1,\ldots,k$ and every $r=0,1,\ldots,\mu_j-1$. Clearly (5.22) is of the form

$$\sum_{j=1}^{k}p_j(t)\mathrm{e}^{\lambda_j t}=0, \tag{5.23}$$

where, for $j=1,\ldots,k$,

$$p_j(t)=\sum_{r=0}^{\mu_j-1}\alpha_{jr}t^r$$

is a polynomial of degree μ_j-1. Suppose that for some $j=1,\ldots,k$ and $r=0,1,\ldots,\mu_j-1$, the constant α_{jr} is non-zero. Then clearly the polynomial $p_j(t)$ cannot be identically zero. Without loss of generality assume that $p_k(t)$ is not identically zero. Multiplying (5.23) by $\mathrm{e}^{-\lambda_1 t}$, we have

$$p_1(t)+\sum_{j=2}^{k}p_j(t)\mathrm{e}^{(\lambda_j-\lambda_1)t}=0. \tag{5.24}$$

Recall that $p_1(t)$ is a polynomial of degree μ_1-1. It follows that on differentiating (5.24) μ_1 times with respect to t, we have

$$\sum_{j=2}^{k}q_j(t)\mathrm{e}^{(\lambda_j-\lambda_1)t}=0,$$

i.e.

$$\sum_{j=2}^{k}q_j(t)\mathrm{e}^{\lambda_j t}=0.$$

Note that for every $j=2,\ldots,k$, the polynomial $q_j(t)$ has the same degree as the polynomial $p_j(t)$. Repeating this argument a finite number of times, we get

$$s_k(t)\mathrm{e}^{\lambda_k t}=0. \tag{5.25}$$

Recall now that we have assumed that $p_k(t)$ is not identically zero, so that (5.25) is clearly false.

To end this section, we state

Proposition 5.8. *Suppose that the characteristic polynomial* (5.2) *has real coefficients. Suppose further that it has distinct real roots λ_j, $j=1,\ldots,k$, with multiplicities $\mu_1,\ldots,\mu_k$ respectively, and distinct pairs of non-real roots $\lambda_j^+=u_j+\mathrm{i}v_j$ and $\lambda_j^-=u_j-\mathrm{i}v_j$, $j=k+1,\ldots,s$, with multiplicities $\mu_{k+1},\ldots,\mu_s$ respectively. Then the real valued functions*

$$\mathrm{e}^{\lambda_j t},t\mathrm{e}^{\lambda_j t},\ldots,t^{\mu_j-1}\mathrm{e}^{\lambda_j t},\quad j=1,\ldots,k,$$

together with

$$\mathrm{e}^{u_j t}\cos v_j t,t\mathrm{e}^{u_j t}\cos v_j t,\ldots,t^{\mu_j-1}\mathrm{e}^{u_j t}\cos v_j t,\quad j=k+1,\ldots,s,$$

and

$$\mathrm{e}^{u_j t}\sin v_j t,t\mathrm{e}^{u_j t}\sin v_j t,\ldots,t^{\mu_j-1}\mathrm{e}^{u_j t}\sin v_j t,\quad j=k+1,\ldots,s,$$

form a fundamental system of solutions of the differential equation (5.1).

REMARK. Note that $n = \mu_1 + \ldots + \mu_k + 2(\mu_{k+1} + \ldots + \mu_s)$.

5.4. Non-Homogeneous Equations

Consider the n-th order non-homogeneous linear differential equation

$$z^{(n)} + a_1 z^{(n-1)} + \ldots + a_n z = b(t), \tag{5.26}$$

where $z = z(t)$ is an unknown real or complex valued scalar function, $a_1, \ldots, a_n$ are real or complex constants, and the function $b(t)$ is continuous on (r_1, r_2). Suppose that we have the initial conditions

$$z(t_0) = z_0, \quad z'(t_0) = z_0', \quad \ldots, \quad z^{(n-1)}(t_0) = z_0^{(n-1)}. \tag{5.27}$$

Then as a special case of Proposition 4.6, we have

PROPOSITION 5.9. *Suppose that $b(t)$ is continuous on (r_1, r_2), and that $\psi_1(t), \ldots, \psi_n(t)$ is a fundamental system of solutions of* (5.1). *Suppose further that $t_0 \in (r_1, r_2)$. Then the solution of the equation* (5.26) *with initial conditions* (5.27) *is given by*

$$y(t) = \psi_0(t) + W^{-1}(0) \sum_{j=1}^{n} \psi_j(t) \int_{t_0}^{t} \mathrm{e}^{a_1 s} b(s) W_j(s)\, \mathrm{d}s,$$

where $\psi_0(t)$ is the solution of the homogeneous equation (5.1) *with given initial conditions* (5.27)*, and where, for every $j = 1, \ldots, n$, $W_j(t) = \det \mathcal{W}_j(t)$ and the matrix $\mathcal{W}_j(t)$ is obtained from the matrix $\mathcal{W}(t)$ by replacing the j-th column by $(0, \ldots, 0, 1)$.*

EXAMPLE. Consider the fourth order linear differential equation

$$z'''' - 4z''' + 3z'' + 4z' - 4z = \mathrm{e}^{2t}.$$

The corresponding homogeneous equation has characteristic polynomial $p^4 - 4p^3 + 3p^2 + 4p - 4$, with roots $\lambda_1 = \lambda_2 = 2$, $\lambda_3 = 1$ and $\lambda_4 = -1$. A fundamental system of solutions is therefore

$$\psi_1(t) = \mathrm{e}^{2t}, \quad \psi_2(t) = t\mathrm{e}^{2t}, \quad \psi_3(t) = \mathrm{e}^{t}, \quad \psi_4(t) = \mathrm{e}^{-t},$$

with fundamental matrix

$$\mathcal{W}(t) = \begin{pmatrix} \mathrm{e}^{2t} & t\mathrm{e}^{2t} & \mathrm{e}^{t} & \mathrm{e}^{-t} \\ 2\mathrm{e}^{2t} & (2t+1)\mathrm{e}^{2t} & \mathrm{e}^{t} & -\mathrm{e}^{-t} \\ 4\mathrm{e}^{2t} & (4t+4)\mathrm{e}^{2t} & \mathrm{e}^{t} & \mathrm{e}^{-t} \\ 8\mathrm{e}^{2t} & (8t+12)\mathrm{e}^{2t} & \mathrm{e}^{t} & -\mathrm{e}^{-t} \end{pmatrix}.$$

Suppose now we have the initial conditions $z(0) = 1$, $z'(0) = 2$, $z''(0) = 12$ and $z'''(0) = 30$. Then

$$\psi_0(t) = \mathrm{e}^{2t} + 2t\mathrm{e}^{2t} - \mathrm{e}^{t} + \mathrm{e}^{-t}$$

is the solution of the homogeneous equation satisfying the given initial conditions. On the other hand,

$$W_1(s) = \det \begin{pmatrix} 0 & s\mathrm{e}^{2s} & \mathrm{e}^{s} & \mathrm{e}^{-s} \\ 0 & (2s+1)\mathrm{e}^{2s} & \mathrm{e}^{s} & -\mathrm{e}^{-s} \\ 0 & (4s+4)\mathrm{e}^{2s} & \mathrm{e}^{s} & \mathrm{e}^{-s} \\ 1 & (8s+12)\mathrm{e}^{2s} & \mathrm{e}^{s} & -\mathrm{e}^{-s} \end{pmatrix} = 2\mathrm{e}^{2s}(3s+4),$$

$$W_2(s) = \det \begin{pmatrix} \mathrm{e}^{2s} & 0 & \mathrm{e}^{s} & \mathrm{e}^{-s} \\ 2\mathrm{e}^{2s} & 0 & \mathrm{e}^{s} & -\mathrm{e}^{-s} \\ 4\mathrm{e}^{2s} & 0 & \mathrm{e}^{s} & \mathrm{e}^{-s} \\ 8\mathrm{e}^{2s} & 1 & \mathrm{e}^{s} & -\mathrm{e}^{-s} \end{pmatrix} = -6\mathrm{e}^{2s},$$

$$W_3(s) = \det \begin{pmatrix} \mathrm{e}^{2s} & s\mathrm{e}^{2s} & 0 & \mathrm{e}^{-s} \\ 2\mathrm{e}^{2s} & (2s+1)\mathrm{e}^{2s} & 0 & -\mathrm{e}^{-s} \\ 4\mathrm{e}^{2s} & (4s+4)\mathrm{e}^{2s} & 0 & \mathrm{e}^{-s} \\ 8\mathrm{e}^{2s} & (8s+12)\mathrm{e}^{2s} & 1 & -\mathrm{e}^{-s} \end{pmatrix} = -9\mathrm{e}^{3s},$$

$$W_4(s) = \det \begin{pmatrix} \mathrm{e}^{2s} & s\mathrm{e}^{2s} & \mathrm{e}^{s} & 0 \\ 2\mathrm{e}^{2s} & (2s+1)\mathrm{e}^{2s} & \mathrm{e}^{s} & 0 \\ 4\mathrm{e}^{2s} & (4s+4)\mathrm{e}^{2s} & \mathrm{e}^{s} & 0 \\ 8\mathrm{e}^{2s} & (8s+12)\mathrm{e}^{2s} & \mathrm{e}^{s} & 1 \end{pmatrix} = \mathrm{e}^{5s}.$$

It follows that

$$\begin{aligned}
&\sum_{j=1}^{4}\psi_j(t)\int_0^t \mathrm{e}^{a_1 s}b(s)W_j(s)\,\mathrm{d}s \\
&\quad = \mathrm{e}^{2t}\int_0^t \mathrm{e}^{-2s}2\mathrm{e}^{2s}(3s+4)\,\mathrm{d}s + t\mathrm{e}^{2t}\int_0^t \mathrm{e}^{-2s}(-6\mathrm{e}^{2s})\,\mathrm{d}s + \mathrm{e}^{t}\int_0^t \mathrm{e}^{-2s}(-9\mathrm{e}^{3s})\,\mathrm{d}s + \mathrm{e}^{-t}\int_0^t \mathrm{e}^{-2s}\mathrm{e}^{5s}\,\mathrm{d}s \\
&\quad = \left(-3t^2+8t-\frac{26}{3}\right)\mathrm{e}^{2t}+9\mathrm{e}^{t}-\frac{1}{3}\mathrm{e}^{-t}.
\end{aligned}$$

Furthermore,

$$W(0)=\det\begin{pmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 1 & -1 \\ 4 & 4 & 1 & 1 \\ 8 & 12 & 1 & -1 \end{pmatrix} = -18.$$

Hence

$$\begin{aligned}
z(t) &= \mathrm{e}^{2t}+2t\mathrm{e}^{2t}-\mathrm{e}^{t}+\mathrm{e}^{-t}+\left(\frac{1}{6}t^2-\frac{4}{9}t+\frac{13}{27}\right)\mathrm{e}^{2t}-\frac{1}{2}\mathrm{e}^{t}+\frac{1}{54}\mathrm{e}^{-t} \\
&= \mathrm{e}^{2t}+2t\mathrm{e}^{2t}-\frac{3}{2}\mathrm{e}^{t}+\frac{55}{54}\mathrm{e}^{-t}+\left(\frac{1}{6}t^2-\frac{4}{9}t+\frac{13}{27}\right)\mathrm{e}^{2t}
\end{aligned}$$

is the solution of the equation with given initial condition.

5.5. Method of Undetermined Coefficients

Let us return to equation (5.26).

Suppose that $b(t)$ is a solution of some homogeneous linear equation with constant coefficients,

$$z^{(m)}+b_1z^{(m-1)}+\ldots+b_mz=0,$$

say. Then writing $M(D)=D^m+b_1D^{m-1}+\ldots+b_m$, we have

$$M(D)b(t)=0. \tag{5.28}$$

If we write $L(D)=D^n+a_1D^{n-1}+\ldots+a_n$, then (5.26) becomes

$$L(D)z=b(t). \tag{5.29}$$

Combining (5.28) and (5.29), we obtain

$$M(D)L(D)z=0. \tag{5.30}$$

Writing $N(D)=M(D)L(D)$, (5.30) can be rewritten in the form

$$N(D)z=0. \tag{5.31}$$

Note now that (5.31) is a homogeneous linear equation with constant coefficients and degree $r=n+m$.

Let $\chi_1(t),\ldots,\chi_r(t)$ be a fundamental system of solutions of (5.31). Then the solution of (5.31) is given by

$$\chi(t)=\sum_{i=1}^{r}\beta_i\chi_i(t), \tag{5.32}$$

where $\beta_1,\ldots,\beta_r$ are arbitrary constants. Substituting this into (5.29), we obtain

$$L(D)\chi(t)=b(t). \tag{5.33}$$

Next, note that if $\varphi_1(t),\ldots,\varphi_n(t)$ is a fundamental system of solutions of $L(D)z=0$, then clearly $\varphi_1(t),\ldots,\varphi_n(t)$ form a linearly independent set of solutions of (5.31). Without loss of generality, we may therefore assume that

$$\chi_i(t)=\varphi_i(t), \quad i=1,\ldots,n. \tag{5.34}$$

If we now combine (5.32) and (5.33), and noting (5.34) and

$$L(D)\varphi_i(t)=0, \quad i=1,\ldots,n,$$

we obtain

$$b(t) = \sum_{i=1}^{r} \beta_i L(D)\chi_i(t) = \sum_{i=n+1}^{r} \beta_i L(D)\chi_i(t). \tag{5.35}$$

We can now determine $\beta_{n+1}, \ldots, b_r$ by comparing coefficients on the two sides of (5.35). Note also that

$$\sum_{i=1}^{n} \beta_i \chi_i(t)$$

is a solution of the homogeneous equation $L(D)z = 0$, and the coefficients $\beta_1, \ldots, \beta_n$ are determined by initial conditions.

Example. Consider the differential equation

$$z'' - z = \mathrm{e}^{2t}(t^2 + 1). \tag{5.36}$$

This can be expressed in the form

$$L(D)z = \mathrm{e}^{2t}(t^2 + 1),$$

where $L(D) = D^2 - 1$. The function $b(t) = \mathrm{e}^{2t}(t^2 + 1)$ satisfies the homogeneous linear equation $M(D)b(t) = 0$, where $M(D) = (D - 2)^3$. In our notation, we consider the equation $N(D)z = 0$, where $N(D) = (D - 2)^3(D^2 - 1)$. A fundamental system of solutions of $N(D)z = 0$ is given by

$$\chi_1(t) = \mathrm{e}^{t}, \quad \chi_2(t) = \mathrm{e}^{-t}, \quad \chi_3(t) = \mathrm{e}^{2t}, \quad \chi_4(t) = t\mathrm{e}^{2t}, \quad \chi_5(t) = t^2\mathrm{e}^{2t}.$$

Hence

$$\chi(t) = \beta_1\mathrm{e}^{t} + \beta_2\mathrm{e}^{-t} + \beta_3\mathrm{e}^{2t} + \beta_4 t\mathrm{e}^{2t} + \beta_5 t^2\mathrm{e}^{2t}$$

is a solution of (5.36) for suitable choices of the coefficients. Now

$$\begin{aligned} L(D)\chi(t) &= \beta_3 L(D)\mathrm{e}^{2t} + \beta_4 L(D)t\mathrm{e}^{2t} + \beta_5 L(D)t^2\mathrm{e}^{2t} \\ &= (3\beta_5 t^2 + (8\beta_5 + 3\beta_4)t + (2\beta_5 + 4\beta_4 + 3\beta_3))\mathrm{e}^{2t}. \end{aligned}$$

But $L(D)\chi(t) = \mathrm{e}^{2t}(t^2 + 1)$, so we must have

$$t^2 + 1 = 3\beta_5 t^2 + (8\beta_5 + 3\beta_4)t + (2\beta_5 + 4\beta_4 + 3\beta_3).$$

Comparing coefficients, we have

$$\beta_3 = \frac{35}{27}, \quad \beta_4 = -\frac{8}{9}, \quad \beta_5 = \frac{1}{3}.$$

The solution of (5.36) is now given by

$$\chi(t) = \beta_1\mathrm{e}^{t} + \beta_2\mathrm{e}^{-t} + \mathrm{e}^{2t}\left(\frac{1}{3}t^2 - \frac{8}{9}t + \frac{35}{27}\right).$$

Note that β_1 and β_2 are determined by initial conditions.

5.6. Behaviour of Solutions

Consider the n-th order homogeneous linear equation

$$z^{(n)} + a_1 z^{(n-1)} + \ldots + a_n z = 0, \tag{5.37}$$

where $a_1, \ldots, a_n$ are real or complex constants. The solutions $z(t)$ are defined on $t \in (-\infty, \infty)$. In practical situations, we are interested in the behaviour of the solutions as $t \to +\infty$.

If the characteristic polynomial of the equation has distinct roots $\lambda_1, \ldots, \lambda_k$, with multiplicities $\mu_1, \ldots, \mu_k$ respectively, then by Proposition 5.7, any solution $z(t)$ is a linear combination of the functions

$$\mathrm{e}^{\lambda_1 t}, t\mathrm{e}^{\lambda_1 t}, \ldots, t^{\mu_1 - 1}\mathrm{e}^{\lambda_1 t}, \quad \ldots, \quad \mathrm{e}^{\lambda_k t}, t\mathrm{e}^{\lambda_k t}, \ldots, t^{\mu_k - 1}\mathrm{e}^{\lambda_k t}.$$

Let $\xi(t) = t^r\mathrm{e}^{\lambda t}$ be one of these functions. If λ has negative real part, then clearly

$$\lim_{t \to +\infty} t^r\mathrm{e}^{\lambda t} = 0.$$

We have proved

PROPOSITION 5.10. *Suppose that all the roots of the characteristic polynomial of the equation* (5.37) *have negative real parts. Then given any solution* $z(t)$ *of* (5.37), *we have*

$$\lim_{t\to+\infty} z(t) = 0.$$

Next, consider a function of the form $\xi(t) = \mathrm{e}^{\lambda t}$. If λ has non-positive real part, then clearly $\mathrm{e}^{\lambda t}$ is bounded as $t \to +\infty$. We have therefore proved the following variation to Proposition 5.10.

PROPOSITION 5.11. *Suppose that all the roots of the characteristic polynomial of the equation* (5.37) *have non-positive real parts, and that all such roots with multiplicity greater than* 1 *have negative real parts. Then any solution* $z(t)$ *of* (5.37) *remains bounded as* $t \to +\infty$.

These are examples of situations when we say the solutions of the system are stable.

Problems for Chapter 5

1. For each of the following differential equations, solve the corresponding homogeneous equation. Then solve the non-homogeneous equation using the method of undetermined coefficients, without guessing the particular integral:

(i) $z'' - 3z' + 2z = \cos t$
(ii) $z'' - 3z' + 2z = \mathrm{e}^t$
(iii) $z'' - 3z' + 2z = \mathrm{e}^t + 2t\mathrm{e}^{2t}$
(iv) $z''' - z' = t^2\mathrm{e}^t$
(v) $z'' + 4z' + 3z = t\sin t + \mathrm{e}^t$

2. Consider the Euler equation

$$t^n z^{(n)} + a_1 t^{n-1} z^{(n-1)} + \ldots + a_n z = 0,$$

where $a_1, \ldots, a_n$ are constants.

(i) Show that the substitution $t = \mathrm{e}^u$ reduces the equation to an n-th order linear equation with constant coefficients. Describe a fundamental system of solutions of equation.
(ii) Consider the case $n = 2$. Find conditions on the coefficients a_1 and a_2 to ensure that all solutions of the equation are bounded as $t \to +\infty$.

3. Solve the following Euler equations using the method in Question 2(i):

(i) $t^2 z'' - 4tz' + 6z = 0$
(ii) $t^2 z'' - 3tz' + 5z = 0$
(iii) $t^3 z''' + tz' - z = 0$