CHAPTER 6

First Order Linear Systems with Constant Coefficients

6.1. Homogeneous Systems

Consider the first order homogeneous linear system

$$x' = Ax. \tag{6.1}$$

Here A is a constant $n \times n$ matrix with real or complex entries, and $x = x(t) = (x_1(t), \ldots, x_n(t))$ is an unknown vector function, considered as an n-dimensional column vector.

It was shown in Chapter 3 that given any initial condition

$$x(t_0) = x_0 = (x_{10}, \ldots, x_{n0}), \tag{6.2}$$

there is a unique solution to the system (6.1) satisfying (6.2). It was also shown that a fundamental system of solutions $\varphi_1(t), \ldots, \varphi_n(t)$ of (6.1) exists. Here in the special case when A is constant, we shall attempt to describe such a fundamental system.

Suppose that B is a constant invertible $n \times n$ matrix. Write $y = B^{-1}x$. Then $x = By$. Furthermore, $x' = By'$. It follows that (6.1) can be described by $By' = ABy$, *i.e.*

$$y' = B^{-1}ABy. \tag{6.3}$$

The idea is to choose B so that $B^{-1}AB$ has simple form.

Consider the polynomial $\det(A - pI) = 0$. This can be written in the form

$$(p - \lambda_1)^{\mu_1} \ldots (p - \lambda_k)^{\mu_k} = 0, \tag{6.4}$$

where the distinct roots $\lambda_1, \ldots, \lambda_k \in \mathbb{C}$ are the eigenvalues, with multiplicities $\mu_1, \ldots, \mu_k$ respectively. It is well known that we can choose B so that $B^{-1}AB$ is in Jordan normal form. When this is achieved, we can then study the system (6.3) in Section 6.2. The reduction of the matrix A to Jordan normal form is covered in any standard course in linear algebra. However, for the sake of completeness, we briefly discuss this in Section 6.3.

6.2. A Fundamental System of Solutions

Suppose now that we have found an invertible matrix B such that $J = B^{-1}AB$ is in Jordan normal form; in other words,

$$J = \begin{pmatrix} J_1 & & & \\ & J_2 & & \\ & & \ddots & \\ & & & J_k \end{pmatrix},$$

where, for every $j = 1, \ldots, k$, the $\mu_j \times \mu_j$ matrix

$$J_j = \begin{pmatrix} \lambda_j & v_2 & & & \\ & \lambda_j & v_3 & & \\ & & \lambda_j & \ddots & \\ & & & \ddots & v_{\mu_j} \\ & & & & \lambda_j \end{pmatrix},$$

with $v_2, \ldots, v_{\mu_j} \in \{0, 1\}$, and $\lambda_1, \ldots, \lambda_k$ are distinct eigenvalues of the matrix A, with multiplicities $\mu_1, \ldots, \mu_k$ respectively. This means that

$$J = \begin{pmatrix} J_1^* & & & \\ & J_2^* & & \\ & & \ddots & \\ & & & J_m^* \end{pmatrix}, \tag{6.5}$$

where for every $k = 1, \ldots, m$, the matrix J_k^* is of the form

$$\begin{pmatrix} \lambda & 1 & & & \\ & \lambda & 1 & & \\ & & \lambda & \ddots & \\ & & & \ddots & 1 \\ & & & & \lambda \end{pmatrix} \tag{6.6}$$

for some eigenvalue λ of A and some $\mu \leqslant n$. Note that the numbers λ may no longer be distinct.

Consider first of all the first order μ-dimensional system

$$y' = Ky, \tag{6.7}$$

where the matrix K is of the form (6.6).

LEMMA 6.1. *A fundamental system of solutions of the first order μ-dimensional system* (6.7) *is given by*

$$\begin{aligned}
\phi_1[\lambda, \mu](t) &= (\mathrm{e}^{\lambda t}, \underbrace{0, \ldots, 0}_{\mu-1}), \\
\phi_2[\lambda, \mu](t) &= (t\mathrm{e}^{\lambda t}, \mathrm{e}^{\lambda t}, \underbrace{0, \ldots, 0}_{\mu-2}), \\
\phi_3[\lambda, \mu](t) &= \left(\frac{1}{2!}t^2\mathrm{e}^{\lambda t}, t\mathrm{e}^{\lambda t}, \mathrm{e}^{\lambda t}, \underbrace{0, \ldots, 0}_{\mu-3}\right), \\
\phi_4[\lambda, \mu](t) &= \left(\frac{1}{3!}t^3\mathrm{e}^{\lambda t}, \frac{1}{2!}t^2\mathrm{e}^{\lambda t}, t\mathrm{e}^{\lambda t}, \mathrm{e}^{\lambda t}, \underbrace{0, \ldots, 0}_{\mu-4}\right), \\
&\vdots \\
\phi_\mu[\lambda, \mu](t) &= \left(\frac{1}{(\mu-1)!}t^{\mu-1}\mathrm{e}^{\lambda t}, \ldots, \frac{1}{2!}t^2\mathrm{e}^{\lambda t}, t\mathrm{e}^{\lambda t}, \mathrm{e}^{\lambda t}\right).
\end{aligned}$$

PROOF. It is not difficult to check for every $j = 1, \ldots, \mu$, the vector $\phi_j[\lambda, \mu](t)$ is a solution of the system (6.7). Note also that the Wronskian $W(0) = 1$. ◯

Consider now of all the first order n-dimensional system

$$y' = Jy, \tag{6.8}$$

where the matrix J is of the form (6.5), and where, for every $k = 1, \ldots, m$, the matrix J_k^* is of the form (6.6) for some eigenvalue λ_k of A and some $\mu_k \leqslant n$.

LEMMA 6.2. *A fundamental system of solutions of the first order μ-dimensional system* (6.8) *is given by*

$$
\begin{aligned}
&(\phi_1[\lambda_1,\mu_1](t), \underbrace{0, \quad \ldots, \quad 0}_{\mu_2+\ldots+\mu_m}),\\
&\quad\vdots\\
&(\phi_{\mu_1}[\lambda_1,\mu_1](t), \underbrace{0, \quad \ldots, \quad 0}_{\mu_2+\ldots+\mu_m});\\
&\quad\vdots\\
&(\underbrace{0, \quad \ldots, \quad 0}_{\mu_1+\ldots+\mu_{k-1}}, \phi_1[\lambda_k,\mu_k](t), \underbrace{0, \quad \ldots, \quad 0}_{\mu_{k+1}+\ldots+\mu_m}),\\
&\quad\vdots\\
&(\underbrace{0, \quad \ldots, \quad 0}_{\mu_1+\ldots+\mu_{k-1}}, \phi_{\mu_k}[\lambda_k,\mu_k](t), \underbrace{0, \quad \ldots, \quad 0}_{\mu_{k+1}+\ldots+\mu_m});\\
&\quad\vdots\\
&(\underbrace{0, \quad \ldots, \quad 0}_{\mu_1+\ldots+\mu_{m-1}}, \phi_1[\lambda_m,\mu_m](t)),\\
&\quad\vdots\\
&(\underbrace{0, \quad \ldots, \quad 0}_{\mu_1+\ldots+\mu_{m-1}}, \phi_{\mu_m}[\lambda_m,\mu_m](t)).
\end{aligned}
$$

PROOF. It is not difficult to check for every vector given is a solution of the system (6.8), in view of Lemma 6.1. Note again that the Wronskian $W(0) = 1$. ◯

We now return to the system (6.1).

LEMMA 6.3. *Suppose that $\psi_1(t), \ldots, \psi_n(t)$ form a fundamental system of solutions of the system* (6.8)*, where $J = B^{-1}AB$ is in Jordan normal form. Then $B\psi_1(t), \ldots, B\psi_n(t)$ form a fundamental system of solutions of the system* (6.1)*.*

PROOF. For every $k = 1, \ldots, n$, since $\psi_k'(t) = J\psi_k(t)$, it follows that $B\psi_k'(t) = BJB^{-1}B\psi_k(t)$. Note now that $A = BJB^{-1}$. On the other hand, the Wronskian of $B\psi_1(t), \ldots, B\psi_n(t)$ is the product of the determinant of B and the Wronskian of $\psi_1(t), \ldots, \psi_n(t)$, and so non-zero. ◯

To summarize, given any first order homogeneous linear system $x' = Ax$, we first reduce the matrix A to Jordan normal form J. We then find a fundamental system of solutions of the system $y' = Jy$ by using Lemma 6.2. Finally, we obtain a fundamental system of solutions of the original system in view of Lemma 6.3.

EXAMPLE. Consider the 3-dimensional system $x' = Ax$, where

$$A = \begin{pmatrix} -3 & 1 & -1 \\ -7 & 5 & -1 \\ -6 & 6 & -2 \end{pmatrix}.$$

If

$$B = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix},$$

then it is not difficult to check that

$$B^{-1}AB = J = \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 4 \end{pmatrix}.$$

We now need to solve the 3-dimensional system $y' = Jy$. By Lemma 6.2, a fundamental system of solutions of $y' = Jy$ is given by

$$\begin{aligned}\psi_1(t) &= (\mathrm{e}^{-2t}, 0, 0),\\ \psi_2(t) &= (t\mathrm{e}^{-2t}, \mathrm{e}^{-2t}, 0),\\ \psi_3(t) &= (0, 0, \mathrm{e}^{4t}).\end{aligned}$$

By Lemma 6.3, a fundamental system of solutions of $x' = Ax$ is given by

$$\begin{aligned}B\psi_1(t) &= (\mathrm{e}^{-2t}, \mathrm{e}^{-2t}, 0),\\ B\psi_2(t) &= (t\mathrm{e}^{-2t} + \mathrm{e}^{-2t}, t\mathrm{e}^{-2t} + \mathrm{e}^{-2t}, -\mathrm{e}^{-2t}),\\ B\psi_3(t) &= (0, \mathrm{e}^{4t}, \mathrm{e}^{4t}).\end{aligned}$$

6.3. Jordan Normal Form

This section is concerned with describing the linear algebra necessary to reduce the matrix A to Jordan normal form, and may be skipped if you are familiar with Jordan normal form.

LEMMA 6.4. *Let A be an $n \times n$ matrix with characteristic polynomial* (6.4)*. For every $j = 1, \dots, k$, let $V_j = \{u \in \mathbb{C}^n : (A - \lambda_j I)^{\mu_j} u = 0\}$. Then $\mathbb{C}^n$ is the direct sum of $V_1, \dots, V_k$.*

PROOF. (i) We shall first of all show that $\mathbb{C}^n$ is the direct sum of V_1 and W, where

$$W = \{u \in \mathbb{C}^n : (A - \lambda_2 I)^{\mu_2} \dots (A - \lambda_k I)^{\mu_k} u = 0\}.$$

To do this, let

$$f_1(p) = (p - \lambda_1)^{\mu_1} \quad \text{and} \quad f_2(p) = (p - \lambda_2)^{\mu_2} \dots (p - \lambda_k)^{\mu_k}.$$

Then $f_1(p)$ and $f_2(p)$ are coprime, so there exist polynomials $g_1(p)$ and $g_2(p)$, with coefficients in $\mathbb{C}$, such that $g_1(p)f_1(p) + g_2(p)f_2(p) = 1$. Hence

$$g_1(A)f_1(A) + g_2(A)f_2(A) = I. \tag{6.9}$$

Let $u \in \mathbb{C}^n$. Then

$$u = g_1(A)f_1(A)u + g_2(A)f_2(A)u.$$

Now $g_1(A)f_1(A)u \in W$, since

$$f_2(A)g_1(A)f_1(A)u = g_1(A)f_1(A)f_2(A)u = 0.$$

Similarly $g_2(A)f_2(A)u \in V_1$. Hence $\mathbb{C}^n$ is a sum of V_1 and W. To show that $\mathbb{C}^n$ is the direct sum of V_1 and W, we shall show that for every $u \in \mathbb{C}^n$, the expression

$$u = v_1 + w, \quad v_1 \in V, \ w \in W,$$

is unique. Note first of all that since $f_1(A)v_1 = 0$, we must have

$$g_1(A)f_1(A)u = g_1(A)f_1(A)w. \tag{6.10}$$

On the other hand, using (6.9) and noting that $f_2(A)w = 0$, we must have

$$w = g_1(A)f_1(A)w. \tag{6.11}$$

Combining (6.10) and (6.11), we conclude that $w = g_1(A)f_1(A)u$. Similarly $v_1 = g_2(A)f_2(A)u$.

(ii) We now assume that W is a direct sum of $V_2', \dots, V_k'$, where for every $j = 2, \dots, k$,

$$V_j' = \{u \in W : (A - \lambda_j I)^{\mu_j} u = 0\}.$$

Clearly $V_j' \subseteq V_j$. On the other hand, suppose that $u \in V_j$. Then $(A - \lambda_j I)^{\mu_j} u = 0$, so that $(A - \lambda_2 I)^{\mu_2} \dots (A - \lambda_k I)^{\mu_k} u = 0$, whence $u \in W$. Hence $V_j \subseteq V_j'$. ○

LEMMA 6.5. *Let V be a vector space over $\mathbb{C}$, of dimension μ. Suppose that $f : V \to V$ is a linear map, and that there exists an integer $r \geqslant 0$ such that $f^r = 0$ on V. Then there exists a basis of V with respect to which the matrix of f is of the form*

$$\begin{pmatrix} 0 & v_2 & & & & \\ & 0 & v_3 & & & \\ & & 0 & \ddots & & \\ & & & \ddots & v_\mu \\ & & & & 0 \end{pmatrix},$$

where $v_2, \ldots, v_\mu \in \{0, 1\}$.

To prove Lemma 6.5, we may assume that $f \neq 0$ on V, otherwise the result is trivial. Hence there exists an integer $q \geqslant 1$ such that

$$f^q \neq 0 \quad \text{and} \quad f^{q+1} = 0$$

on V. For every integer $r \geqslant 0$, let

$$E_r = \ker f^r.$$

Clearly $E_0 = \{0\}$ and $E_{q+1} = V$.

LEMMA 6.6. *We have $\{0\} = E_0 \subsetneqq E_1 \subsetneqq \ldots \subsetneqq E_q \subsetneqq E_{q+1} = V$. Furthermore, $f(E_{i+1}) \subseteq E_i$ for every $i = 0, 1, \ldots, q$.*

LEMMA 6.7. *Let $i = 1, \ldots, q$ be chosen. Suppose that W is a subspace of V such that $W \cap E_i = \{0\}$. Then $f(W) \cap E_{i-1} = \{0\}$. Furthermore, f induces an isomorphism of W onto $f(W)$.*

LEMMA 6.8. *There exist subspaces $W_1, \ldots, W_{q+1}$ of V such that*

(i) *for every $i = 1, \ldots, q + 1$, E_i is the direct sum of E_{i-1} and W_i; and*
(ii) *for every $i = 2, \ldots, q + 1$, f maps W_i one-to-one into W_{i-1}.*

PROOF OF LEMMA 6.5. We first of all construct the subspaces $W_1, \ldots, W_{q+1}$ of V as in Lemma 6.8. Let

$$v_{1,1}, v_{1,2}, \ldots, v_{1,r_1}$$

be a basis of W_{q+1}. Since these vectors are linearly independent and since f maps W_{q+1} one-to-one into W_q, the vectors

$$f(v_{1,1}), f(v_{1,2}), \ldots, f(v_{1,r_1})$$

are linearly independent. We can extend this collection to a basis of W_q of the form

$$v_{2,1}, \ldots, v_{2,r_1}, v_{2,r_1+1}, \ldots, v_{2,r_2},$$

where

$$f(v_{1,j}) = v_{2,j}, \quad j = 1, \ldots, r_1.$$

Repeating this argument, we can show that there exists a basis of W_{q-1} of the form

$$v_{3,1}, \ldots, v_{3,r_2}, v_{3,r_2+1}, \ldots, v_{3,r_3},$$

where

$$f(v_{2,j}) = v_{3,j}, \quad j = 1, \ldots, r_2.$$

Continuing in this way, we finally arrive at a basis of $W_1 = E_1$ of the form

$$v_{q+1,1}, \ldots, v_{q+1,r_{q+1}}, \tag{6.12}$$

where

$$f(v_{q,j}) = v_{q+1,j}, \quad j = 1, \ldots, r_q.$$

Since $E_1 = \ker f$, we also have

$$f(v_{q+1,j}) = 0, \quad j = 1, \ldots, r_{q+1}.$$

It is easy to see from Lemma 6.8(i) that V is the direct sum of $W_1, \ldots, W_{q+1}$, and a basis can be constructed using the vectors we have discussed so far. We can write the elements of this basis in the following order:

$$\begin{array}{ccccccccccc}
v_{1,1} & \ldots & v_{1,r_1} & & & & & & & & \\
v_{2,1} & \ldots & v_{2,r_1} & v_{2,r_1+1} & \ldots & v_{2,r_2} & & & & & \\
\vdots & & \vdots & \vdots & & \vdots & & & & & \\
v_{q+1,1} & \ldots & v_{q+1,r_1} & v_{q+1,r_1+1} & \ldots & v_{q+1,r_2} & \ldots & v_{q+1,r_q+1} & \ldots & v_{q+1,r_{q+1}}
\end{array}$$

The elements can be rewritten in the following order: column by column from left to right, and each column from bottom to top, and denoted by

$$v_i, \quad i = 1, \ldots, r_1 + \ldots + r_{q+1}.$$

Then either $f(v_i) = 0$ or $f(v_i) = v_{i-1}$, and the matrix of f with respect to this basis has the form indicated in the statement of Lemma 6.5. ◯

Proof of Lemma 6.6. The second assertion follows from the definition of E_r. To prove the first assertion, note that clearly $E_i \subseteq E_{i+1}$. On the other hand, suppose on the contrary that $E_i = E_{i+1}$ for some $i = 0, 1, \ldots, q$. Then for every $v \in V$, $0 = f^{q+1}(v) = f^{i+1}(f^{q-i}(v))$, so that $f^{q-i}(v) \in E_{i+1} = E_i$, whence $f^q(v) = 0$, contradicting the definition of q. ◯

Proof of Lemma 6.7. Let $v \in f(W) \cap E_{i-1}$. Then there exists $w \in W$ such that $v = f(w)$, so that $0 = f^{i-1}(v) = f^i(w)$, whence $w \in E_i$. Hence $w = 0$, and so $v = 0$. This proves the first assertion. On the other hand, the mapping $f|_W : W \to f(W)$ is clearly linear and onto. It is easy to check that it is also one-to-one. ◯

Proof of Lemma 6.8. Choose W_{q+1} so that $E_{q+1} = V$ is a direct sum of E_q and W_{q+1}. Then $f(W_{q+1}) \subseteq E_q$. By Lemma 6.7, $f(W_{q+1}) \cap E_{q-1} = \{0\}$. Hence there exists W_q such that $f(W_{q+1}) \subseteq W_q$ and such that E_q is a direct sum of E_{q-1} and W_q. We continue this way to construct $W_{q-1}, \ldots, W_1$, where $f(W_{i+1}) \subseteq W_i$ for every $i = 1, \ldots, q$. Note that (ii) follows from the second assertion of Lemma 6.7. ◯

To obtain Jordan normal form, we first of all apply Lemma 6.4. Consider the restriction of A to the elements of each V_j. Then $A - \lambda_j I$ can be described by a matrix of the form given in Lemma 6.5. Obtain a Jordan bases for V_j. We now take the union of all such bases obtained for $V_1, \ldots, V_k$. Jordan normal form follows easily.