CHAPTER 7

Series Solutions of Second Order Linear Equations

7.1. Introduction

Suppose that $w = w(z)$ is a complex valued function of a complex variable z. We are interested in a differential equation of the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + p(z)\frac{\mathrm{d}w}{\mathrm{d}z} + q(z)w = 0, \tag{7.1}$$

where $p(z)$ and $q(z)$ are given complex valued functions. The purpose of this chapter is to attempt to find a solution of (7.1) of the form

$$w(z) = (z - z_0)^r \sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{7.2}$$

where the power series in (7.2) converges in some neighbourhood of the point $z = z_0$.

Recall that a complex valued function $f(z)$ of a complex variable z is said to be analytic at $z = z_0$ if $f(z)$ has a Taylor series expansion

$$f(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

which converges in some neighbourhood of $z = z_0$. Recall also that a complex valued function $f(z)$ of a complex variable z is said to have a pole of order $k \in \mathbb{N}$ at $z = z_0$ if $g(z) = f(z)(z - z_0)^k$ is analytic at $z = z_0$ and $g(z_0) \neq 0$.

7.2. The Singular Case

Let us return to the differential equation (7.1).

Definition. We say that a point $z = z_0$ is a singular point of the differential equation (7.1) if $p(z)$ or $q(z)$ is not analytic at $z = z_0$. Furthermore, we say that the singular point $z = z_0$ is regular if the following extra conditions are satisfied:

(i) if $p(z)$ has a pole at $z = z_0$, then the pole is of order 1; and
(ii) if $q(z)$ has a pole at $z = z_0$, then the pole is of order 1 or 2.

It follows that if $z = z_0$ is a regular singular point of the equation (7.1), then (7.1) can be rewritten in the form

$$\frac{\mathrm{d}^2 w}{\mathrm{d}z^2} + \frac{P(z)}{(z - z_0)}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{Q(z)}{(z - z_0)^2} w = 0 \tag{7.3}$$

where the functions $P(z)$ and $Q(z)$ are analytic at $z = z_0$. Write

$$P(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n \quad \text{and} \quad Q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n, \tag{7.4}$$

where the series (7.4) converge for $|z - z_0| < a$, where $a > 0$.

Proposition 7.1. *Suppose that $z = z_0$ is a regular singular point of the differential equation (7.1), so that (7.1) can be rewritten in the form (7.3), and the series (7.4) converges for $|z - z_0| < a$, where $a > 0$. Then there exists a solution of (7.1) of the form (7.2), valid in a punctured neighbourhood of $z = z_0$.*

Let us substitute the expressions (7.2) and (7.4) into (7.3). Then

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(z-z_0)^{n+r-2}&\\ +(z-z_0)^{-1}\left(\sum_{n=0}^{\infty} p_n(z-z_0)^n\right)&\left(\sum_{n=0}^{\infty} a_n(n+r)(z-z_0)^{n+r-1}\right)\\ +(z-z_0)^{-2}\left(\sum_{n=0}^{\infty} q_n(z-z_0)^n\right)&\left(\sum_{n=0}^{\infty} a_n(z-z_0)^{n+r}\right)=0.\end{aligned}$$

Multiplying by $(z-z_0)^{2-r}$, we have

$$\begin{aligned}\sum_{n=0}^{\infty} a_n(n+r)(n+r-1)(z-z_0)^n + \left(\sum_{n=0}^{\infty} p_n(z-z_0)^n\right)\left(\sum_{n=0}^{\infty} a_n(n+r)(z-z_0)^n\right)&\\ + \left(\sum_{n=0}^{\infty} q_n(z-z_0)^n\right)\left(\sum_{n=0}^{\infty} a_n(z-z_0)^n\right)=0.&\end{aligned}\tag{7.5}$$

Note that the left hand side of (7.5) is analytic in $|z-z_0| < a$ if the series (7.2) converges in $|z-z_0| < a$. Take any $n \in \mathbb{N}\cup\{0\}$. Then the coefficient of $(z-z_0)^n$ on the left hand side of (7.5) is

$$a_n(n+r)(n+r-1) + \sum_{k=0}^{n} p_k a_{n-k}(n+r-k) + \sum_{k=0}^{n} q_k a_{n-k} = 0.\tag{7.6}$$

Write

$$G(n+r) = (n+r)^2 + (p_0-1)(n+r) + q_0.$$

Then the left hand side of (7.6) becomes

$$a_n G(n+r) + H_{n,r},$$

where

$$H_{n,r} = H_{n,r}(a_0,\ldots,a_{n-1},p_1,\ldots,p_n,q_1,\ldots,q_n) = \sum_{k=1}^{n} p_k a_{n-k}(n+r-k) + \sum_{k=1}^{n} q_k a_{n-k}.\tag{7.7}$$

Furthermore, in the case $n = 0$, (7.6) becomes $a_0 G(r) = 0$. If we choose any $a_0 \neq 0$, then

$$G(r) = r^2 + (p_0-1)r + q_0 = 0.\tag{7.8}$$

The equation (7.8) is known as the indicial equation of (7.3) at $z = z_0$. Suppose that its roots are given by r_1 and r_2. Assume, without loss of generality, that $\mathfrak{Re}(r_1 - r_2) \geqslant 0$. Since $G(r_1) = 0$ and $p_0 - 1 = -(r_1 + r_2) = -2r_1 + (r_1 - r_2)$, it follows that for all $n \in \mathbb{N}$,

$$G(n+r_1) - G(r_1) = (n+r_1)^2 + (p_0-1)(n+r_1) - r_1^2 - (p_0-1)r_1 = n^2 + 2nr_1 + (p_0-1)n,$$

so that

$$G(n+r_1) = G(r_1) + n^2 + 2nr_1 + (p_0-1)n = n^2 + n(p_0 - 1 + 2r_1) = n(n + r_1 - r_2) \neq 0$$

since $|n + r_1 - r_2| \geqslant n$. With this fixed value r_1, we can now solve the recurrence equations

$$a_n G(n+r_1) + H_n = 0\tag{7.9}$$

for a_n in terms of $a_0,\ldots,a_{n-1},p_1,\ldots,p_n,q_1,\ldots,q_n$, where $H_n = H_{n,r_1}$. It follows that (7.2) with $r = r_1$ represents a formal solution of (7.3), and hence of (7.1). Naturally we may choose $a_0 = 1$ if we so wish.

Proof of Proposition 7.1. Suppose that we have carried out the calculation above with the solution r_1 of the indicial equation. Then it remains to show that the series

$$\sum_{n=1}^{\infty} a_n (z - z_0)^n \tag{7.10}$$

has positive radius of convergence. The functions $p(z)$ and $q(z)$ are analytic in $0 < |z - z_0| < a$, so that the solution (7.2) is valid and defined in $0 < |z - z_0| < a$. Now choose $\alpha \in (0, a)$. Then by the Convergence theorem for power series, both

$$\sum_{n=1}^{\infty} |p_n| \alpha^n \quad \text{and} \quad \sum_{n=1}^{\infty} |q_n| \alpha^n$$

converge, so that $|p_n|\alpha^n$ and $|q_n|\alpha^n$ are bounded. It follows that there exists $K > 1$ such that

$$\max\{|p_n|, |r_1 p_n + q_n|, |a_0(r_1 p_n + q_n)|\} < K\alpha^{-n}$$

for every $n \in \mathbb{N}$. We shall prove by induction on n that

$$|a_n| < K^n \alpha^{-n} \quad \text{for all } n \in \mathbb{N}. \tag{7.11}$$

For $n = 1$, note that by (7.7)–(7.9),

$$|a_1| = \left| \frac{H_1}{G(1 + r_1)} \right| = \left| \frac{a_0(p_1 r_1 + q_1)}{1 + r_1 - r_2} \right| < K\alpha^{-1}.$$

Suppose now that $|a_n| < K^n \alpha^{-n}$ for every $n = 1, \ldots, m - 1$. Then

$$\begin{aligned}
|a_m| &= \left| \frac{H_m}{G(m + r_1)} \right| \\
&\leqslant \frac{\sum_{k=1}^{m} |a_{m-k}||r_1 p_k + q_k| + \sum_{k=1}^{m} (m - k)|a_{m-k}||p_k|}{m|m + r_1 - r_2|} \\
&\leqslant \sum_{k=1}^{m} K^{m-k} \alpha^{k-m} K\alpha^{-k} + \sum_{k=1}^{m} (m - k) K^{m-k} \alpha^{k-m} K\alpha^{-k} m^2 \\
&= \left(\frac{\sum_{k=1}^{m} K^{1-k} + \sum_{k=1}^{m} (m - k) K^{1-k}}{m^2} \right) K^m \alpha^{-m} \\
&\leqslant \left(\frac{\sum_{k=1}^{m} 1 + \sum_{k=1}^{m} (m - k)}{m^2} \right) K^m \alpha^{-m} \\
&= \left(\frac{m^2 + m}{2m^2} \right) K^m \alpha^{-m} \leqslant K^m \alpha^{-m}.
\end{aligned}$$

It now follows from (7.11) that if $0 < \beta < \alpha/K$, then for every $n \in \mathbb{N}$, we have

$$|a_n|\beta^n \leqslant K^n \alpha^{-n} \beta^n = \left(\frac{K\beta}{\alpha} \right)^n.$$

Clearly

$$\sum_{n=1}^{\infty} (K\beta/\alpha)^n$$

converges, so that

$$\sum_{n=1}^{\infty} a_n \beta^n$$

converges absolutely. It follows that the series (7.10) has radius of convergence at least α/K. ○

Proposition 7.1 gives rise to one solution of the equation (7.1). To determine a fundamental system of solutions of (7.1), we need to find another solution. To do this, we try to make use of the root r_2 of the indicial equation (7.8). In most instances, this is a simple exercise, as is evident from the following result.

PROPOSITION 7.2. *Suppose that $z = z_0$ is a regular singular point of the differential equation* (7.1)*, so that* (7.1) *can be rewritten in the form* (7.3)*, and the series* (7.4) *converges for $|z - z_0| < a$, where $a > 0$. Suppose further that r_1 and r_2 are the two roots of the indicial equation* (7.8)*, with $\mathfrak{Re}(r_1 - r_2) \geqslant 0$. If $r_1 - r_2 \notin \mathbb{N} \cup \{0\}$, then there exists a second solution of* (7.1) *of the form*

$$w(z) = (z - z_0)^{r_2} \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

valid in a punctured neighbourhood of $z = z_0$ and with $b_0 \neq 0$.

PROOF. Since $r_1 - r_2 \notin \mathbb{N} \cup \{0\}$, it is clear that

$$G(n + r_2) = n^2 + n(p_0 - 1 + 2r_2) = n^2 - n(r_1 - r_2) \neq 0,$$

so that we can solve the recurrence equations

$$a_n G(n + r_2) + H_n = 0.$$

The proof now proceeds as in the proof of Proposition 7.1. ○

Note that if $r_1 - r_2 = 0$, then $r_1 = r_2$, so we clearly cannot follow our argument in Proposition 7.2. If $r_1 - r_2 \in \mathbb{N}$, then let $r_1 - r_2 = m$. Then $G(m + r_2) = m^2 - m(r_1 - r_2) = 0$, so we cannot solve the recurrence equations that arise. It follows that much more work needs to be done if $r_1 - r_2 \in \mathbb{N} \cup \{0\}$. This case is summarized by the following result.

PROPOSITION 7.3. *Suppose that $z = z_0$ is a regular singular point of the differential equation* (7.1)*, so that* (7.1) *can be rewritten in the form* (7.3)*, and the series* (7.4) *converges for $|z - z_0| < a$, where $a > 0$. Suppose further that r_1 and r_2 are the two roots of the indicial equation* (7.8)*, with $\mathfrak{Re}(r_1 - r_2) \geqslant 0$. If $r_1 - r_2 \in \mathbb{N} \cup \{0\}$, then there exists a second solution of* (7.1) *of the form*

$$w(z) = w_1(z)\beta \log(z - z_0) + (z - z_0)^{r_2} \sum_{n=0}^{\infty} b_n (z - z_0)^n,$$

valid in a punctured neighbourhood of $z = z_0$. Here β is a constant, and $w_1(z)$ is the solution given by Proposition 7.1 corresponding to the root r_1 of the indicial equation (7.8)*. Furthermore, $\beta \neq 0$ if $r_1 = r_2$.*

To find a second solution, we now use the method of reduction of order, and try for a solution of the form

$$w_2(z) = w_1(z) \int^z \phi(u)\, \mathrm{d}u, \tag{7.12}$$

where the function $\phi(u)$ will be determined as the solution of a first order differential equation. Here $w_1(z)$ is the solution given by Proposition 7.1 corresponding to the root r_1 of the indicial equation (7.8).

Substituting (7.12) into (7.3), we obtain

$$\begin{aligned}\left(w_1(z)\phi'(z) + 2w_1'(z)\phi(z) + w_1''(z)\int^z \phi(u)\,\mathrm{d}u\right)& \\ + \frac{P(z)}{(z - z_0)}\left(w_1(z)\phi(z) + w_1'(z)\int^z \phi(u)\,\mathrm{d}u\right) + \frac{Q(z)}{(z - z_0)^2}&\left(w_1(z)\int^z \phi(u)\,\mathrm{d}u\right) = 0.\end{aligned} \tag{7.13}$$

Since $w_1(z)$ is a root of (7.3), the equation (7.13) can now be reduced to the form

$$w_1(z)\phi'(z) + 2w_1'(z)\phi(z) + \frac{P(z)}{(z - z_0)} w_1(z)\phi(z) = 0,$$

i.e.

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} + \left(\frac{P(z)}{z - z_0} + \frac{2w_1'(z)}{w_1(z)}\right)\phi = 0. \tag{7.14}$$

Note now that

$$\frac{P(z)}{z - z_0} = \frac{p_0}{z - z_0} + \sum_{n=1}^{\infty} p_n (z - z_0)^{n-1},$$

and

$$\sum_{n=1}^{\infty} p_n (z - z_0)^{n-1}$$

is analytic at $z = z_0$. On the other hand, note that $w_1(z) = (z - z_0)^{r_1} f(z)$, where $f(z)$ is analytic at $z = z_0$ and $f(z_0) \neq 0$. Hence

$$\frac{w_1'(z)}{w_1(z)} = \frac{(z - z_0)^{r_1} f'(z) + r_1 (z - z_0)^{r_1 - 1} f(z)}{(z - z_0)^{r_1} f(z)} = \frac{f'(z)}{f(z)} + \frac{r_1}{z - z_0}.$$

Clearly $f'(z)/f(z)$ is analytic at $z = z_0$. It now follows that

$$\frac{P(z)}{z - z_0} + \frac{2w_1'(z)}{w_1(z)} = \frac{p_0 + 2r_1}{z - z_0} + g(z),$$

where $g(z)$ is analytic at $z = z_0$.

We now solve the equation (7.14) for ϕ. Clearly we can separate the variables ϕ and z, and it is easy to see that a solution is given by

$$\begin{aligned} \phi(z) &= \exp\left(-\int^z \left(\frac{p_0 + 2r_1}{u - z_0} + g(u)\right) \mathrm{d}u\right) \\ &= \exp(-(p_0 + 2r_1)\log(z - z_0)) \exp\left(-\int^z g(u)\, \mathrm{d}u\right) \\ &= (z - z_0)^{-(p_0 + 2r_1)} h(z) = (z - z_0)^{-(r_1 - r_2) - 1} h(z), \end{aligned}$$

where $h(z)$ is analytic at $z = z_0$. Write

$$h(z) = \sum_{n=0}^{\infty} \beta_n (z - z_0)^n.$$

Then since $r_1 - r_2 \in \mathbb{N} \cup \{0\}$, we have

$$\phi(z) = \beta_{r_1 - r_2} (z - z_0)^{-1} + (z - z_0)^{-(r_1 - r_2) - 1} \sum_{\substack{n=0 \\ n \neq r_1 - r_2}}^{\infty} \beta_n (z - z_0)^n.$$

Hence

$$\int^z \phi(u)\, \mathrm{d}u = \beta \log(z - z_0) + (z - z_0)^{-(r_1 - r_2)} k(z),$$

where $\beta = \beta_{r_1 - r_2}$ and where $k(z)$ is analytic at $z = z_0$. We therefore have

$$\begin{aligned} w_2(z) &= w_1(z) \int^z \phi(u)\, \mathrm{d}u \\ &= w_1(z) \beta \log(z - z_0) + (z - z_0)^{-(r_1 - r_2)} k(z) (z - z_0)^{r_1} \sum_{n=0}^{\infty} a_n (z - z_0)^n \\ &= w_1(z) \beta \log(z - z_0) + (z - z_0)^{r_2} \ell(z), \end{aligned}$$

where

$$\ell(z) = \sum_{n=0}^{\infty} b_n (z - z_0)^n$$

is analytic at $z = z_0$. Note finally that it is possible that $\beta = 0$. However, if $r_1 = r_2$, then clearly $\beta = \beta_0 = h(z_0) \neq 0$ (why?).

7.3. The Analytic Case

Let us again return to the differential equation (7.1).

PROPOSITION 7.4. *Suppose that the functions $p(z)$ and $q(z)$ are analytic at $z = z_0$. Then there exists a solution of* (7.1) *of the form*

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n, \tag{7.15}$$

valid in a neighbourhood of $z = z_0$.

Since $p(z)$ and $q(z)$ are analytic at $z = z_0$, we can write

$$p(z) = \sum_{n=0}^{\infty} p_n (z - z_0)^n \quad \text{and} \quad q(z) = \sum_{n=0}^{\infty} q_n (z - z_0)^n. \tag{7.16}$$

Subsitiuting (7.15) and (7.16) into (7.1), we obtain

$$\sum_{n=2}^{\infty} n(n-1) a_n (z - z_0)^{n-2} + \left(\sum_{n=0}^{\infty} p_n (z - z_0)^n \right) \left(\sum_{n=1}^{\infty} n a_n (z - z_0)^{n-1} \right)$$
$$+ \left(\sum_{n=0}^{\infty} q_n (z - z_0)^n \right) \left(\sum_{n=0}^{\infty} a_n (z - z_0)^n \right) = 0,$$

i.e.

$$\sum_{n=0}^{\infty} (n+2)(n+1) a_{n+2} (z - z_0)^n + \left(\sum_{n=0}^{\infty} p_n (z - z_0)^n \right) \left(\sum_{n=0}^{\infty} (n+1) a_{n+1} (z - z_0)^n \right) \tag{7.17}$$
$$+ \left(\sum_{n=0}^{\infty} q_n (z - z_0)^n \right) \left(\sum_{n=0}^{\infty} a_n (z - z_0)^n \right) = 0.$$

Then the coefficient of $(z - z_0)^n$ on the left hand side of (7.17) is

$$(n+2)(n+1) a_{n+2} + \sum_{k=0}^{n} (k+1) a_{k+1} p_{n-k} + \sum_{k=0}^{n} a_k q_{n-k} = 0. \tag{7.18}$$

It follows that if we are given a_0 and a_1, determined by given initial conditions, we can then solve the recurrence equations (7.18) for $a_2, a_3, \ldots$ in terms of the given values of a_0 and a_1.

PROOF OF PROPOSITION 7.4. Suppose that we have carried out the calculation above with given a_0 and a_1. Then it remains to show that the series (7.15) has positive radius of convergence. Note that since the functions $p(z)$ and $q(z)$ are analytic at $z = z_0$, the series (7.16) have positive radius of convergence a, say. Now choose $\alpha \in (0, a)$. Then by the Convergence theorem for power series, both

$$\sum_{n=0}^{\infty} |p_n| \alpha^n \quad \text{and} \quad \sum_{n=0}^{\infty} |q_n| \alpha^n$$

converge, so that $|p_n| \alpha^n$ and $|q_n| \alpha^n$ are bounded. It follows that there exists $K > 1$ such that

$$\max\{|p_n|, |q_n|\} < K \alpha^{-n}$$

for every $n \in \mathbb{N} \cup \{0\}$. Furthermore, we may assume without loss of generality that

$$K > |a_0| \quad \text{and} \quad \alpha < 1. \tag{7.19}$$

We shall prove by induction on n that

$$|a_n| < K^n \alpha^{-n} \quad \text{for all } n \in \mathbb{N}. \tag{7.20}$$

For $n = 1$, we simply have to make sure that K is chosen to be sufficiently large and α is assumed to be sufficiently small. Suppose now that $|a_n| < K^n \alpha^{-n}$ for every $n = 1, \ldots, m+1$. Then it follows from (7.18) and (7.19) that

$$\begin{aligned}(m+2)(m+1)|a_{m+2}| &\leqslant \sum_{k=0}^{m}(k+1)K^{k+1}\alpha^{-k-1}K\alpha^{k-m} + \sum_{k=0}^{m}K^{k+1}\alpha^{-k}K\alpha^{k-m} \\ &= \sum_{k=0}^{m}(k+1)K^{k+2}\alpha^{-m-1} + \sum_{k=0}^{m}K^{k+2}\alpha^{-m} \\ &< K^{m+2}\alpha^{-m-1}\sum_{k=0}^{m}(k+1) + K^{m+2}\alpha^{-m}\sum_{k=0}^{m}1 \\ &= K^{m+2}\alpha^{-m-1}\frac{(n+1)(n+2)}{2} + (n+1)K^{m+2}\alpha^{-m},\end{aligned}$$

so that

$$|a_{m+2}| < \left(\frac{\alpha}{2} + \frac{\alpha^2}{m+2}\right)K^{m+2}\alpha^{-m-2} < K^{m+2}\alpha^{-m-2}.$$

It now follows from (7.20) that if $0 < \beta < \alpha/K$, then for every $n \in \mathbb{N}$, we have

$$|a_n|\beta^n \leqslant K^n\alpha^{-n}\beta^n = \left(\frac{K\beta}{\alpha}\right)^n.$$

Clearly

$$\sum_{n=1}^{\infty}\left(\frac{K\beta}{\alpha}\right)^n$$

converges, so that

$$\sum_{n=0}^{\infty}a_n\beta^n$$

converges absolutely. It follows that the series (7.15) has radius of convergence at least α/K. ○

7.4. An Example

To illustrate the singular case, consider the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + \frac{z}{z+1}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{z+1}w = 0$$

around the regular singular point $z = -1$. In the notation of Section 7.2, we have

$$P(z) = -1 + (z+1) \quad \text{and} \quad Q(z) = z+1,$$

so that $p_0 = -1$, $p_1 = 1$, $p_2 = p_3 = \ldots = 0$ and $q_0 = 0$, $q_1 = 1$, $q_2 = q_3 = \ldots = 0$. The indicial equation (7.8) is of the form $r^2 - 2r = 0$, with roots $r_1 = 2$ and $r_2 = 0$. Corresponding to $r_1 = 2$, we now assume a solution of the form

$$w_1(z) = (z+1)^2\sum_{n=0}^{\infty}a_n(z+1)^n.$$

Then the recurrence equations (7.9) are of the form

$$n(n+2)a_n = -\sum_{k=1}^{n}(n+2-k)p_ka_{n-k} - \sum_{k=1}^{n}q_ka_{n-k} = -(n+2)a_{n-1},$$

so that $na_n = -a_{n-1}$. It follows that if we take $a_0 = 1$, then $a_n = (-1)^n/n!$ for every $n \in \mathbb{N}$. It follows that

$$w_1(z) = (z+1)^2\sum_{n=0}^{\infty}\frac{(-1)^n}{n!}(z+1)^n = (z+1)^2\mathrm{e}^{-z-1}. \tag{7.21}$$

Next, we assume a second solution of the form

$$w_2(z) = w_1(z)\int^z\phi(u)\,\mathrm{d}u,$$

where $w_1(z)$ is given by (7.21). Then equation (7.14) is of the form

$$\frac{\mathrm{d}\phi}{\mathrm{d}z} + \left(\frac{z}{z+1} + \frac{4}{z+1} - 2\right)\phi = 0.$$

This has a solution given by $\phi(z) = \mathrm{e}^z(z+1)^{-3}$, leading to a second solution of the form

$$w_2(z) = \frac{1}{2}(z+1)^2\mathrm{e}^{-z-1}\log(z+1) + \sum_{n=0}^{\infty} b_n(z+1)^n.$$

To illustrate the analytic case, consider the same differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + \frac{z}{z+1}\frac{\mathrm{d}w}{\mathrm{d}z} + \frac{1}{z+1}w = 0$$

around the point $z=0$, where both functions $p(z)$ and $q(z)$ are analytic. In the notation of Section 7.3, we have $p(z) = 1-(1+z)^{-1} = z - z^2 + z^3 - z^4 + \ldots$ and $q(z) = (1+z)^{-1} = 1 - z + z^2 - z^3 + \ldots$, so that $p_0 = 0$ and $p_n = (-1)^{n-1}$ for every $n \in \mathbb{N}$, and $q_n = (-1)^n$ for every $n \in \mathbb{N}\cup\{1\}$. Now suppose that $a_0 = 0$ and $a_1 = 1$. Then

$$\begin{aligned}(n+2)(n+1)a_{n+2} &= -\sum_{k=0}^{n-1}(k+1)(-1)^{n-k-1}a_{k+1} - \sum_{k=1}^{n}(-1)^{n-k}a_k \\ &= -\sum_{k=0}^{n-1}(k+1)(-1)^{n-k-1}a_{k+1} - \sum_{k=0}^{n-1}(-1)^{n-k-1}a_{k+1} \\ &= \sum_{k=0}^{n-1}(k+2)(-1)^{n-k}a_{k+1}.\end{aligned} \tag{7.22}$$

Note also that $a_2 = 0$. We can now calculate $a_3, a_4, \ldots$. Here that the calculation can be somewhat simplified if we note that

$$(n+3)(n+2)a_{n+3} = \sum_{k=0}^{n}(k+2)(-1)^{n+1-k}a_{k+1}. \tag{7.23}$$

Adding (7.22) and (7.23) and then dividing by $(n+2)$, we have

$$(n+3)a_{n+3} = -(n+1)a_{n+2} - a_{n+1}$$

for every $n \in \mathbb{N}\cup\{0\}$. This gives $a_3 = -1/3$, $a_4 = 1/6$, $a_5 = -1/30$, *etc.*

Problems for Chapter 7

1. For each of the following cases, find two linearly independent solutions of the differential equation

$$a_0(z)\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + a_1(z)\frac{\mathrm{d}w}{\mathrm{d}z} + a_2(z)w = 0$$

near the point $z = 0$:

(i) $a_0(z) = 2z^2$, $a_1(z) = -z$, $a_2(z) = z^2 + 1$
(ii) $a_0(z) = z$, $a_1(z) = 2$, $a_2(z) = z^2$
(iii) $a_0(z) = z$, $a_1(z) = 1$, $a_2(z) = -z$
(iv) $a_0(z) = 2z^2$, $a_1(z) = -z$, $a_2(z) = 1 - z^2$
(v) $a_0(z) = z$, $a_1(z) = z - 1$, $a_2(z) = -1$
(vi) $a_0(z) = z$, $a_1(z) = 1$, $a_2(z) = z^2$
(vii) $a_0(z) = z^2$, $a_1(z) = 3z$, $a_2(z) = 1 + z$
(viii) $a_0(z) = z^2$, $a_1(z) = z$, $a_2(z) = z^2 - \alpha^2$, where $\alpha \notin \mathbb{N} \cup \{0\}$
[*Remark*: This is known as Bessel's equation.]
(ix) $a_0(z) = 1 - z^2$, $a_1(z) = -2z$, $a_2(z) = \alpha(\alpha + 1)$, where α is constant
[*Remark*: This is known as Legendre's equation.]