CHAPTER 8

Existence and Uniqueness of Solutions

8.1. Introduction

Our method of proving the Theorem is known as the method of successive approximations. We choose a first approximation $x_0(t)$ to a solution, using the initial data. We then devise an algorithm and use this to construct successive approximations $x_1(t), x_2(t), \ldots, x_n(t), \ldots$ to a solution. Note here that each $x_n(t)$ may not be a solution of the given system at all. However, we shall show that the sequence of approximations $x_n(t)$ converges, in some suitable sense, to a solution $x(t)$ as $n \to \infty$.

To illustrate the method, let us consider the following example.

EXAMPLE. Note that the function $x = 3\mathrm{e}^t$ satisfies the differential equation

$$x' = x \quad \text{and} \quad x(0) = 3. \tag{8.1}$$

We shall now try to show that $x = 3\mathrm{e}^t$ is the limit of a "convergent" sequence of functions. Since $x' = x$, any solution $x(t)$ must satisfy

$$\int_0^t x'(s)\,\mathrm{d}s = \int_0^t x(s)\,\mathrm{d}s,$$

so that

$$x(t) = 3 + \int_0^t x(s)\,\mathrm{d}s.$$

To construct a sequence $x_n(t)$ to "converge" to a solution $x(t)$, we may perhaps choose $x_n(t)$ to satisfy the recurrence equations

$$x_{n+1}(t) = 3 + \int_0^t x_n(s)\,\mathrm{d}s.$$

Choose, for example, $x_0(t) = 3$. Then it is easy to show by induction on n that

$$x_n(t) = 3\left(1 + t + \frac{t^2}{2!} + \ldots + \frac{t^n}{n!}\right)$$

for every $n \in \mathbb{N} \cup \{0\}$. Note that $x_n(t)$ "converges" to $x(t) = 3\mathrm{e}^t$.

8.2. A Metric Space Setting

We shall now look at the situation more formally. First of all, in order to show that $x_n(t)$ "converges" to $x(t)$, we need a formal definition of the distance between functions, as the convergence cannot depend on any particular choice of t.

Suppose that $n \in \mathbb{N}$ and $a, b \in \mathbb{R}$ satisfying $a < b$ are fixed. Denote by $C[a, b]$ the collection of all n-dimensional vector functions $x(t) = (x_1(t), \ldots, x_n(t))$ which are continuous on $[a, b]$, *i.e.* every $x_j(t)$, $j = 1, \ldots, n$, is continuous on $[a, b]$.

DEFINITION. For every function $x \in C[a,b]$, we define the norm $N(x)$ of x by

$$\|x(t)\| = \sum_{j=1}^{n} |x_j(t)| \quad \text{and} \quad N(x) = \sup_{t\in[a,b]} \|x(t)\|.$$

REMARK. Suppose that $x \in C[a,b]$. Given $t \in [a,b]$, we may think of $\|x(t)\|$, the sum of the moduli of the coordinates of $x(t)$, as the "size" of $x(t)$. Then $N(x)$ can be interpreted as the maximum "size" of $x(t)$ as t runs over $[a,b]$.

DEFINITION. For any two functions $x, y \in C[a,b]$, we define the distance $d(x,y)$ between x and y by $d(x,y) = N(x-y)$.

PROPOSITION 8.1. *The set $C[a,b]$, together with the function $d : C[a,b] \times C[a,b] \to \mathbb{R}$, forms a metric space.*

REMARK. A set C, together with a function $d : C \times C \to \mathbb{R}$, is said to form a metric space, denoted by (C,d), if for every $x, y, z \in C$,

(M1) $d(x,y) \geqslant 0$;
(M2) $d(x,y) = 0$ if and only if $x = y$;
(M3) $d(x,y) = d(y,x)$; and
(M4) $d(x,z) \leqslant d(x,y) + d(y,z)$.

PROOF OF PROPOSITION 8.1. Note that for every $x, y \in C[a,b]$,

$$d(x,y) = N(x-y) = \sum_{j=1}^{n} |x_j(t) - y_j(t)|.$$

(M1), (M2) and (M3) follow easily. To prove (M4), note that if $x, y, z \in C[a,b]$, then by the usual traingle inequality, we have

$$\begin{aligned} d(x,z) = \sum_{j=1}^{n} |x_j(t) - z_j(t)| &\leqslant \sum_{j=1}^{n} (|x_j(t) - y_j(t)| + |y_j(t) - z_j(t)|) \\ &= \sum_{j=1}^{n} |x_j(t) - y_j(t)| + \sum_{j=1}^{n} |y_j(t) - z_j(t)| = d(x,y) + d(y,z), \end{aligned}$$

as required. ○

PROPOSITION 8.2. *The metric space $(C[a,b], d)$ is complete. In other words, every Cauchy sequence in $C[a,b]$ converges to a limit in $C[a,b]$.*

PROOF. Suppose that x_s is a Cauchy sequence in $C[a,b]$. Then

$$\lim_{r,s\to\infty} d(x_r, x_s) = 0.$$

Write $x_r(t) = (y_{r1}(t), \ldots, y_{rn}(t))$ and $x_s(t) = (y_{s1}(t), \ldots, y_{sn}(t))$. Then

$$d(x_r, x_s) = \sup_{t\in[a,b]} \|x_r(t) - x_s(t)\| \geqslant \sup_{t\in[a,b]} |y_{rj}(t) - y_{sj}(t)|$$

for every $j = 1, \ldots, n$, so that

$$\lim_{r,s\to\infty} \sup_{t\in[a,b]} |y_{rj}(t) - y_{sj}(t)| = 0.$$

By the General principle of uniform convergence, the sequence $y_{sj}(t)$ converges uniformly to a function $y_j(t)$. Furthermore, $y_j(t)$ is continuous on $[a,b]$. Let $x(t) = (y_1(t), \ldots, y_n(t))$. Then

$$d(x_s, x) = \sup_{t\in[a,b]} \|x_s(t) - x(t)\| \leqslant \sum_{j=1}^{n} \sup_{t\in[a,b]} |y_{sj}(t) - y_j(t)|,$$

so that

$$\lim_{s\to\infty} d(x_s, x) = 0,$$

as required. ○

Suppose that the real constant $c > 0$ and the point $x_0 \in \mathbb{R}^n$ are fixed. Consider a continuous function $f : E \to \mathbb{R}^n$ defined on the open set

$$E = \{(t, x) : a < t < b \text{ and } \|x - x_0\| < c\}.$$

Suppose that the function $x = x(t)$ is continuous on $[a, b]$, and that the point $(t, x(t)) \in E$ for $r_1 \leqslant t \leqslant r_2$. Suppose further that $r_1 < t_0 < r_2$. Then the function $\mathfrak{A}x : [r_1, r_2] \to \mathbb{R}^n$, defined by

$$\mathfrak{A}x = x_0 + \int_{t_0}^{t} f(s, x(s))\,\mathrm{d}s,$$

is continuous on $[r_1, r_2]$. We may therefore draw the following two conclusions:

(1) We may interpret $\mathfrak{A}$ as a mapping from $C[r_1, r_2]$ into itself.

(2) Since $(\mathfrak{A}x)(t_0) = x_0$, there is a neighbourhood G of t_0 such that $(t, (\mathfrak{A}x)(t)) \in E$ for every $t \in G$.

We now consider the first order n-dimensional system

$$x' = f(t, x). \tag{8.2}$$

Note that any solution $x = x(t)$ defined on $[r_1, r_2]$ and satisfying $x(t_0) = x_0$ must also satisfy

$$x(t) = x_0 + \int_{t_0}^{t} f(s, x(s))\,\mathrm{d}s, \quad t \in [r_1, r_2].$$

In other words, any solution $x = x(t)$ defined on $[r_1, r_2]$ and satisfying $x(t_0) = x_0$ must also satisfy

$$x = \mathfrak{A}x. \tag{8.3}$$

Clearly $x \in C[r_1, r_2]$. Hence the mapping $\mathfrak{A}$ "fixes" any solution $x = x(t)$ of (8.2) defined on $[r_1, r_2]$.

8.3. A Stronger Version of the Theorem

In this section, we shall prove the following slightly stronger version of the Theorem.

Stronger Theorem. *Consider the differential equation*

$$x' = f(t, x), \tag{8.4}$$

where the function $f(t, x)$ is defined on some domain $\mathcal{B} \subseteq \mathbb{R}^{n+1}$. Suppose further that

(i) *f is continuous on $\mathcal{B}$; and*

(ii) *there exists a constant $k > 0$ such that*

$$\|f(t, x_1) - f(t, x_2)\| \leqslant k\|x_1 - x_2\| \tag{8.5}$$

for every pair of points $(t, x_1), (t, x_2) \in \mathcal{B}$.

Then for every point $(t_0, x_0) \in \mathcal{B}$, there exists a unique solution $x = \varphi(t)$ of (8.4) satisfying $x_0 = \varphi(t_0)$ and defined in some neighbourhood of (t_0, x_0).

Remark. The condition (8.5) is usually called a Lipschitz condition.

We now begin the proof of the Stronger Theorem.

Step 1. Since $(t_0, x_0) \in \mathcal{B}$, there exist $a, b > 0$ such that the closed and bounded set

$$\Gamma = \{(t, x) : |t - t_0| \leqslant a \text{ and } \|x - x_0\| \leqslant b\} \subset \mathcal{B}\,.$$

It follows that there exists $m > 0$ such that

$$\|f(t, x)\| \leqslant m \tag{8.6}$$

for all $(t, x) \in \Gamma$.

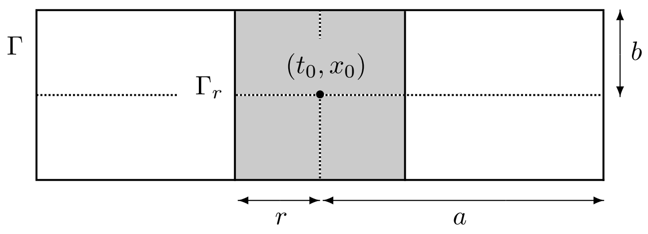
Step 2. Let

$$r \in (0, a], \tag{8.7}$$

to be chosen later, and let

$$\Gamma_r = \{(t, x) : |t - t_0| \leqslant r \text{ and } \|x - x_0\| \leqslant b\} \subset \Gamma. \tag{8.8}$$

Clearly Γ_r is closed and bounded.



STEP 3. Denote by C_r the collection of all functions $x(t) = (x_1(t), \ldots, x_n(t))$ such that

(i) $x(t)$ is continuous on $[t_0 - r, t_0 + r]$, so that $C_r \subseteq C[t_0 - r, t_0 + r]$; and

(ii) $(t, x(t)) \in \Gamma_r$ for every $t \in [t_0 - r, t_0 + r]$, *i.e.* $|t - t_0| \leqslant r$ and $\|x - x_0\| \leqslant b$.

In other words, C_r is a subset of $C[t_0 - r, t_0 + r]$, and the graphs of the functions in C_r all lie in Γ_r.

STEP 4. For every function $x \in C_r$, define the function $\mathfrak{A}x$ by

$$\mathfrak{A}x = x_0 + \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}s \tag{8.9}$$

for every $t \in [t_0 - r, t_0 + r]$.

LEMMA 8.3. *Suppose that*

$$r \leqslant b/m. \tag{8.10}$$

Then $\mathfrak{A}$ is a mapping from C_r into itself. Furthermore, $(\mathfrak{A}x)(t_0) = x_0$.

PROOF. Clearly

$$\begin{aligned} \|\mathfrak{A}x - x_0\| &= \left\| \int_{t_0}^{t} f(s, x(s))\, \mathrm{d}s \right\| = \sum_{j=1}^{n} \left| \int_{t_0}^{t} f_j(s, x(s))\, \mathrm{d}s \right| \leqslant \sum_{j=1}^{n} \int_{t_0}^{t} |f_j(s, x(s))|\, \mathrm{d}s \\ &= \int_{t_0}^{t} \sum_{j=1}^{n} |f_j(s, x(s))|\, \mathrm{d}s = \int_{t_0}^{t} \|f(s, x(s))\|\, \mathrm{d}s \leqslant mr \leqslant b, \end{aligned}$$

in view of (8.6) and (8.10). ◯

STEP 5. For any function $x \in C[t_0 - r, t_0 + r]$, let

$$N(x) = \sup_{t \in [t_0 - r, t_0 + r]} \|x(t)\|.$$

Furthermore, for any functions $x_1, x_2 \in C_r$, write

$$d(x_1, x_2) = N(x_1 - x_2).$$

LEMMA 8.4. *Suppose that*

$$r < 1/k. \tag{8.11}$$

Then there exists $\alpha \in (0, 1)$ such that for any functions $x_1, x_2 \in C_r$,

$$d(\mathfrak{A}x_1, \mathfrak{A}x_2) \leqslant \alpha d(x_1, x_2).$$

In other words, $\mathfrak{A}$ contracts distances in C_r.

PROOF. Note that in view of (8.5),

$$\begin{aligned} \|\mathfrak{A}x_1 - \mathfrak{A}x_2\| &= \left\| \int_{t_0}^{t} (f(s, x_1(s)) - f(s, x_2(s)))\, \mathrm{d}s \right\| \leqslant \int_{t_0}^{t} \|f(s, x_1(s)) - f(s, x_2(s))\|\, \mathrm{d}s \\ &\leqslant k \int_{t_0}^{t} \|x_1(s) - x_2(s)\|\, \mathrm{d}s \leqslant k|t - t_0| \sup_{t \in [t_0 - r, t_0 + r]} \|x_1(s) - x_2(s)\| \\ &\leqslant krd(x_1, x_2). \end{aligned}$$

The result follows if we take $\alpha = kr$. ◯

STEP 6. We now choose r to satisfy (8.7), (8.10) and (8.11) and keep it fixed, and write $\alpha = kr$, so that $\alpha \in (0, 1)$.

STEP 7. Define the function $x_0(t) = x_0$ identically for $t \in [t_0 - r, t_0 + r]$. We then define the sequence of functions x_s inductively by writing $x_s = \mathfrak{A}x_{s-1}$ for every $s \in \mathbb{N}$. Note that the functions $x_0, x_1, x_2, \ldots \in C_r$, and that $x_0(t_0) = x_1(t_0) = x_2(t_0) = \ldots = x_0$.

LEMMA 8.5. *The sequence of functions x_s is a Cauchy sequence in $C[t_0 - r, t_0 + r]$.*

PROOF. Suppose that $s_1, s_2 \in \mathbb{N}$ and $s_1 < s_2$. Then

$$\begin{aligned} d(x_{s_1}, x_{s_2}) &= d(\mathfrak{A}x_{s_1-1}, \mathfrak{A}x_{s_2-1}) \leqslant \alpha d(x_{s_1-1}, x_{s_2-1}) \leqslant \ldots \leqslant \alpha^{s_1} d(x_0, x_{s_2-s_1}) \\ &\leqslant \alpha^{s_1}(d(x_0, x_1) + d(x_1, x_2) + \ldots + d(x_{s_2-s_1-1} - x_{s_2-s_1})) \\ &\leqslant \alpha^{s_1}(d(x_0, x_1) + \alpha d(x_0, x_1) + \ldots + \alpha^{s_2-s_1-1} d(x_0, x_1)) \\ &= \alpha^{s_1} d(x_0, x_1)\left(1 + \alpha + \ldots + \alpha^{s_2-s_1-1}\right) < \frac{\alpha^{s_1}}{1-\alpha} d(x_0, x_1) \to 0 \end{aligned}$$

as $s_1, s_2 \to \infty$. ○

STEP 8. Since $C[t_0 - r, t_0 + r]$ is a complete metric space and x_s is a Cauchy sequence in $C[t_0 - r, t_0 + r]$, it follows from Proposition 8.2 and Lemma 8.3 that there exists a function $x = x(t)$, continuous on $[t_0 - r, t_0 + r]$, such that

(i) $x_s(t) \to x(t)$ uniformly on $[t_0 - r, t_0 + r]$;
(ii) $(t, x(t)) \in \Gamma_r$ for every $t \in [t_0 - r, t_0 + r]$; and
(iii) $x(t_0) = t_0$.

In other words, $x \in C_r$. On the other hand,

$$0 \leqslant d(\mathfrak{A}x, \mathfrak{A}x_s) \leqslant \alpha d(x, x_s) \to 0$$

as $s \to \infty$. Hence

$$\mathfrak{A}x = \lim_{s\to\infty} \mathfrak{A}x_s = \lim_{s\to\infty} x_{s+1} = x.$$

Existence is therefore established.

STEP 9. To prove uniqueness, suppose that $y = y(t)$ is another solution of the differential equation (8.4) defined on $[t_0 - r_1, t_0 + r_1]$, and that $y(t_0) = x_0$. Then there exists $R \leqslant \min\{r, r_1\}$ such that $(t, x(t)), (t, y(t)) \in \Gamma_R$ if $t \in [t_0 - R, t_0 + R]$. Hence

$$d(x, y) = d(\mathfrak{A}x, \mathfrak{A}y) \leqslant \alpha d(x, y).$$

This implies $d(x, y) = 0$, so that $x = y$. Uniqueness is therefore established.