Triangles, Rectangles, Circles and Squares – courtesy of Lionel Salem and Frédéric Testard

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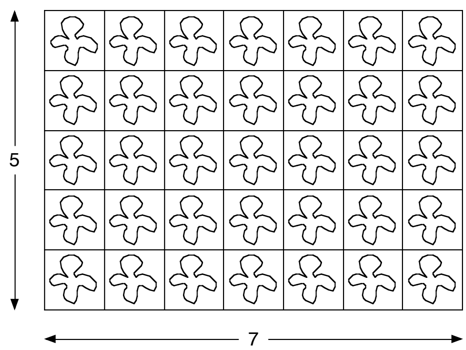
We take a light hearted and amusing look at various basic geometric shapes, using little stories about various imaginary characters. The treatment here follows in part the wonderful book entitled *The Most Beautiful Mathematical Formulas*, written by the chemist Lionel Salem and the mathematician Frédéric Testard, and is in sharp contrast to the traditionally rather algebraic treatment.

Why is the area of a rectangle equal to the product of its sides?

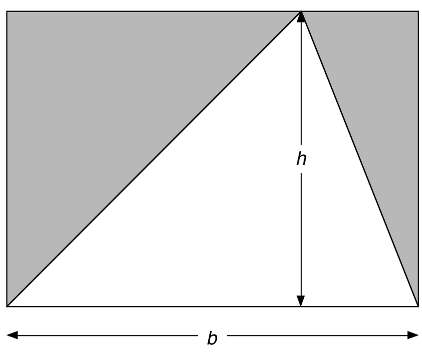
According to Salem and Testard, certain four-leaved clovers have turned into giant plants as the result of a bizarre mutation, and have hastily been baptized *trifolium giganteum* by astonished botanists. These remarkable plants have the strange property of growing only when they occupy a space of one metre by one metre, or one square metre. A farmer who wishes to raise these curious plants happens to have a rectangular field measuring 5 metres by 7 metres. He sees that he can sow 5 rows separated by one metre, with 7 plants in each row.



He therefore cultivates $5 \times 7 = 35$ plants, without wasting any part of his field. The *area*, a term derived from the Latin word for a piece of level ground, is therefore equal to 35 square metres, the product of its length and width.

And how about the area of a triangle?

Growing trifolium gigantea has become such a profitable business that the farmer is making use of every bit of land he has to grow them. The only field he has left is a triangular plot. By another strange genetic twist of fate, the mutant plant will only grow in rectangular fields. Our resourceful farmer has a great idea. He realizes that a triangle is only half a rectangle, and proceeds to cut a deal with his neighbour. Thus he explains, "In my triangular field, I can draw a line h perpendicular to the base b and passing through the opposite vertex. This cuts my field into two right triangles. If we join forces, we can turn the field into a rectangle, since we need only to double each of the triangles."



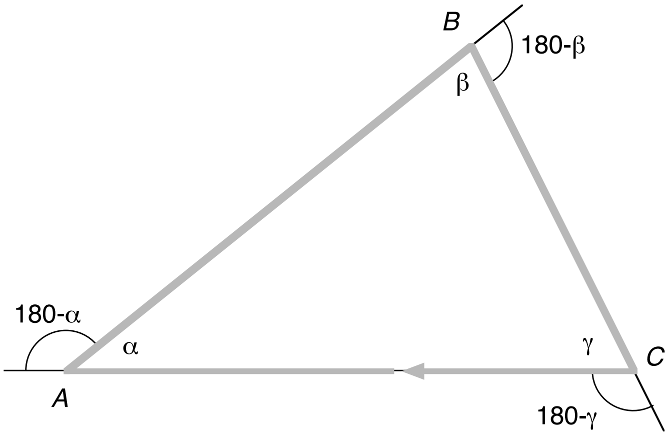
"Since the area of the rectangle is $h \times b$, the area of my triangle must be half of that." Thus the area A of a triangle is given by

$$A = \frac{1}{2}(h \times b),$$

where h is the height and b is the base.

Why is the sum of the angles of a triangle equal to 180 degrees?

One day, the farmer, who has since baptized the three corners of his triangular field A, B and C, is pacing around the perimeter of his triangular plot. He realizes that if the sides AB and AC makes an angle α degrees, then each time he reaches the corner A, he must turn through an angle of $180 - \alpha$ degrees; after all, the sum of the external and internal angles at the corner A is equal to 180 degrees.



Likewise, he turns $180 - \beta$ degrees at the corner B and $180 - \gamma$ degrees at the corner C. After the three turns, he finds himself facing the same direction, having made a complete circuit of 360 degrees. He then realizes that

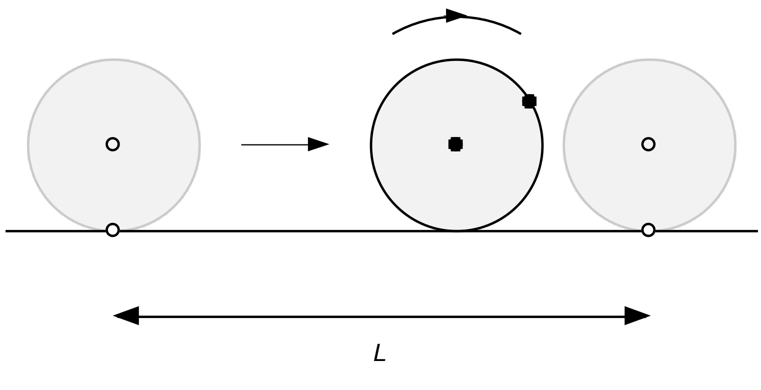
$$(180 - \alpha) + (180 - \beta) + (180 - \gamma) = 360,$$

or that

$$\alpha + \beta + \gamma = 180.$$

Why is the circumference of a circle equal to 2π times its radius?

Growing trifolium gigantea has been sufficiently profitable that our farmer has bought a few bicycles. One day, he notices that it requires more effort to make one full turn of his wheels when their diameter is 70 centimetres than when their diameter is 50 centimetres, and he decides to get to the bottom of this mystery. To measure the length of a full revolution, he takes each of his tyres, puts a little dab of paint on the treads, rotates the tyres and then measures the distance between the spots of paint on the ground.



For the tyre with diameter 50 centimetres, he measures a distance of roughly 1.57 metres, while for the tyre with diameter 70 centimetres, he measures a distance of roughly 2.19 metres. Repeating this experiment with tyres of different diameters, he finds that their circumference L is always proportional to their diameter D, and that the coefficient of proportionality is approximately 3.14; in other words,

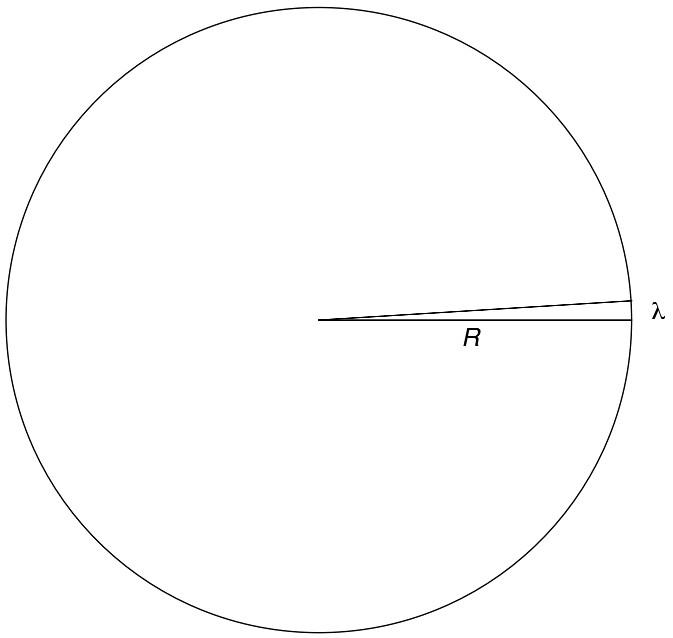
$$L \approx 3.14 \times D.$$

He then discovers in a book on geometry that this coefficient of proportionality is called π, the first letter of the Greek word for *perimeter*. If one replaces the diameter D in this formula by twice the radius R, then the formula becomes

$$L = 2\pi R.$$

Why is the area inside of a circle of radius R equal to πR^2?

Growing trifolium gigantea has been so successful that our farmer decides to have a big celebration and invites everyone in the village. He has arranged for the local baker to prepare a huge round cake. After hours spent cutting it up, each guest receives a small slice.



Some guests realize that the slices resemble right triangles with height R and base λ, as shown in the picture above. The area of each slice is therefore approximately

$$\frac{1}{2}(R \times \lambda).$$

But then one of clever guest points out that if the areas of all the slices are added together, the result will be the radius R times the sum of all the lengths λ divided by 2, which will then be equal to the area A of the cake. He then adds that the sum of all the lengths λ is just the circumference of the cake, which is $2\pi R$. Eureka! We have

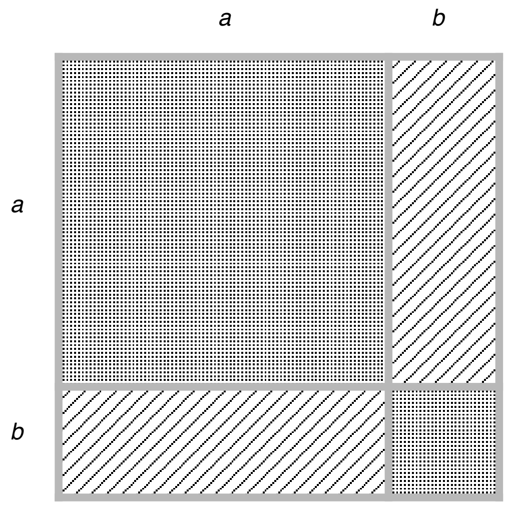
$$A = \frac{1}{2}R \times 2\pi R,$$

or

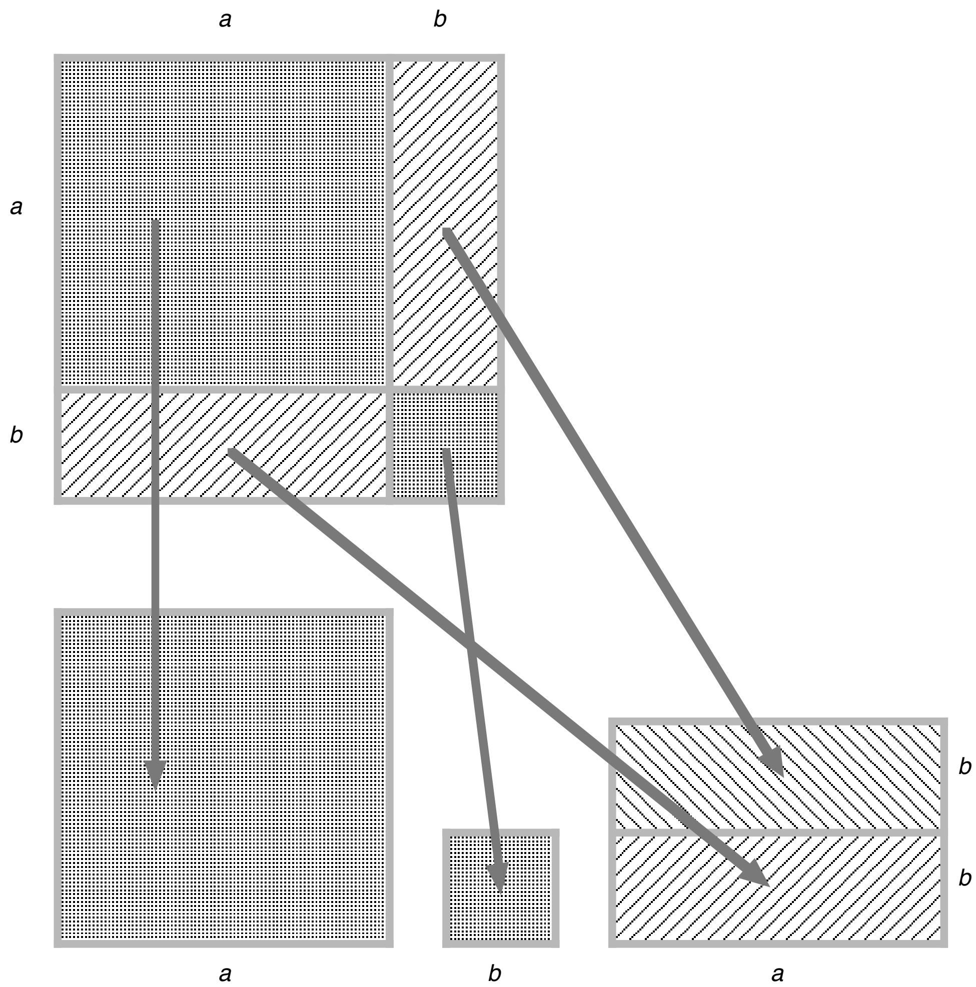
$$A = \pi R^2.$$

How is the algebraic identity $(a+b)^2 = a^2 + 2ab + b^2$ motivated?

One nice day, a smart kid notices that a square poster in the house has some creases that divides the poster into two square subsections, with sides a and b, and two rectangular subsections, with length a and width b.

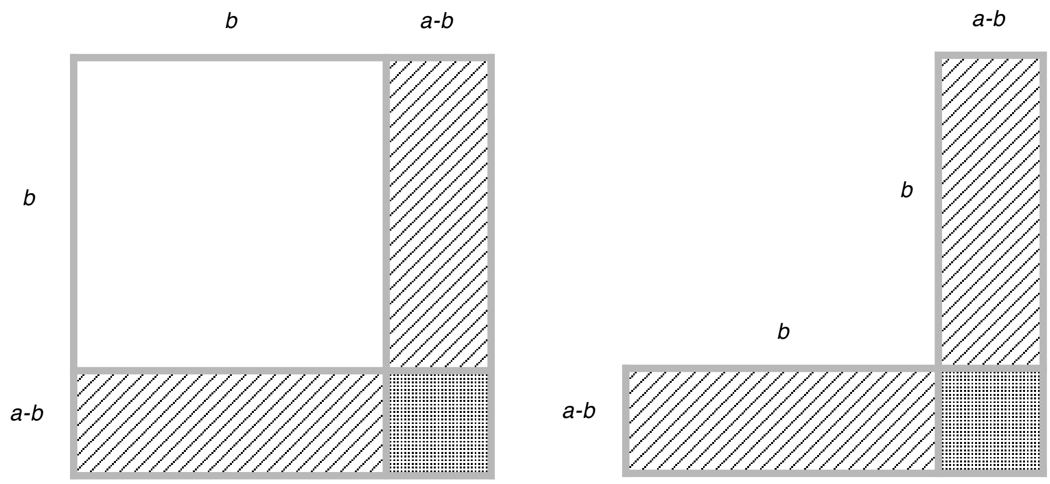


He then decides to calculate the area of the poster in two different ways. First of all, he recognizes that the area must be equal to $(a+b)^2$. But it is also $a^2 + b^2 + ab + ab$. Just to be sure, he decides to cut up the poster along the creases.



And how is the algebraic identity $(a+b)(a-b) = a^2 - b^2$ motivated?

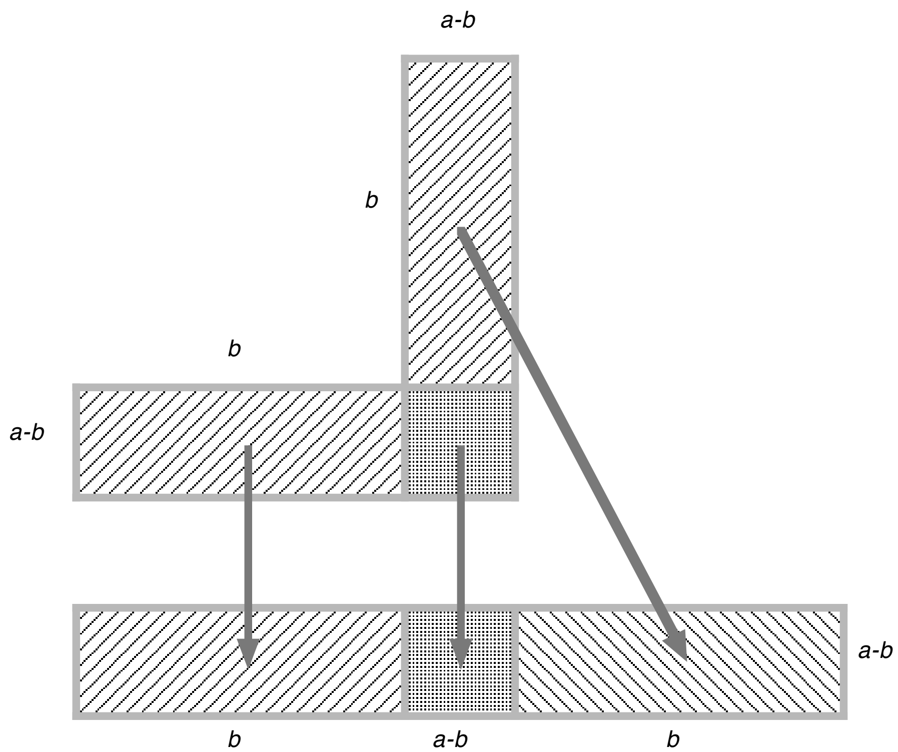
Our smart kid has grown up in the streets and now pastes advertisements on billboards for a living. Having never lost his passion for dabbling in mathematics, he finds many opportunities in his work to put his interest in formulas to good use. From his early experience of cutting up posters, he now has moved on to explore new problems. He starts to hang a square sign with sides a, which has a small square subsection with sides b. He then decides to remove the small square.



This leaves a truncated L-shaped sign which clearly has area

$$a^2 - b^2.$$

Then he moves the upper rectangle with sides b and $a-b$, and places it on the bottom right end of the base.

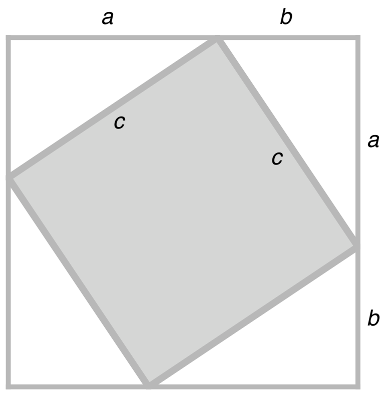


He then notices that this produces a rectangle with length $a+b$ and height $a-b$, and hence area

$$(a+b)(a-b).$$

Now the theorem of Pythagoras, that $a^2 + b^2 = c^2$, where a and b are the two short sides of a right triangle and c is the hypothenuse.

By now, our smart kid has grown up to be a professor. One day, trying to amuse himself, he starts drawing triangles and squares, and comes up with the following beautiful picture of squares and triangles. In particular, it contains right triangles with short sides a and b and hypothenuse c.



He realizes that the big square is made up of the smaller square and four triangles. Now the big square has sides $a + b$, and so the area is $(a + b)^2$. He remembers that as a little boy, cutting up the poster gives

$$(a + b)^2 = a^2 + 2ab + b^2.$$

But then he has also learned from the farmer – remember the one who grows trifolium gigantea – that since each triangle has base a and height b, each has area $\frac{1}{2}ab$, and so the four triangles have total area $2ab$. But now the small square has area c^2, and so the big square has area

$$2ab + c^2.$$

Putting two and two together, our professor now realizes that

$$a^2 + 2ab + b^2 = 2ab + c^2.$$

Removing $2ab$ from each side, the professor realizes that he has rediscovered what Pythagoras showed all those centuries ago, that

$$a^2 + b^2 = c^2.$$

He then spends the rest of his life trying to discover a better proof, without success.