

Binomial Coefficients

William Chen

We briefly make some remarks on a problem that we have considered in the “cat” puzzle, and discuss a simple formula that enables us to perform calculations more easily.

Recall the problem of arranging two letters R and four letters D , where we show that the number of different ways of doing this is 15, simply by writing down all the possibilities. Each of these possibilities corresponds to a different way of choosing 2 boxes out of 6 and placing the letter R in them.



In other words, we are interested in finding the number of ways of choosing 2 objects out of 6.

In general, we have a collection of n objects, and are interested in finding the number of ways of choosing r of them out of these n objects. For instance, we may start with the first n positive integers

$$1, 2, 3, \dots, n,$$

and wish to find out the number of different ways we have of picking out r of them, where $r \leq n$. We denote this number by

$$\binom{n}{r},$$

and this is known as a binomial coefficient “ n choose r ”. You will find that the same quantity is sometimes represented by $C_{n,r}$ or ${}_nC_r$ in some texts. It is defined as follows.

We need first of all to define the factorial of a given positive integer k . This is denoted by $k!$, and is equal to the product of the first k positive integers. In other words, we have

$$k! = 1 \times 2 \times 3 \times \dots \times k.$$

For instance, $1! = 1$, $2! = 2$, $3! = 6$, $4! = 24$ and $5! = 120$. As a convention, we take $0! = 1$; note that under multiplication, if we do not multiply by any number, the result is the same as if we have multiplied by 1.

For any non-negative integer n and any integer r satisfying $0 \leq r \leq n$, we define

$$\binom{n}{r} = \frac{n!}{r! \times (n-r)!}.$$

This is the number of different ways of picking out r objects out of n . Note now that

$$\begin{aligned} \binom{6}{1} &= \frac{6!}{1! \times (6-1)!} = \frac{6!}{1! \times 5!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 1 \times 2 \times 3 \times 4 \times 5} = 6, \\ \binom{6}{2} &= \frac{6!}{2! \times (6-2)!} = \frac{6!}{2! \times 4!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 1 \times 2 \times 3 \times 4} = 15, \\ \binom{6}{3} &= \frac{6!}{3! \times (6-3)!} = \frac{6!}{3! \times 3!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6}{1 \times 2 \times 3 \times 1 \times 2 \times 3} = 20. \end{aligned}$$

Example. Suppose that we have 8 green marbles and 4 blue marbles and wish to line them up from left to right. Then the number of different patterns can be obtained by considering the following analogous problem. Suppose that there are 12 white marbles in a row from left to right.



We now wish to paint 4 of them blue. The number of ways of choosing 4 of these out of 12 is equal to

$$\begin{aligned}\binom{12}{4} &= \frac{12!}{4! \times (12-4)!} = \frac{12!}{4! \times 8!} \\ &= \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12}{1 \times 2 \times 3 \times 4 \times 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \\ &= \frac{9 \times 10 \times 11 \times 12}{1 \times 2 \times 3 \times 4} = 495.\end{aligned}$$

We then simply paint the remaining 8 green.

Example. Suppose that there is an election for 3 positions and there are 8 candidates. Then the number of possible outcomes is equal to

$$\binom{8}{3} = \frac{8!}{3! \times (8-3)!} = \frac{8!}{3! \times 5!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 1 \times 2 \times 3 \times 4 \times 5} = 56.$$

The number of possible outcomes is also equal to

$$\binom{8}{5} = \frac{8!}{5! \times (8-5)!} = \frac{8!}{5! \times 3!} = \frac{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8}{1 \times 2 \times 3 \times 4 \times 5 \times 1 \times 2 \times 3} = 56.$$

Of course, choosing 3 winners out of 8 is the same as choosing 5 losers out of 8.