The Monkey and the Coconuts – courtesy of Ben Ames Williams

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The following problem appeared in the 9 October 1926 issue of the *Sunday Evening Post*, and is reproduced here almost exactly, with only a small change at the end.

"Five men and a monkey were shipwrecked on a desert island, and they spent the first day gathering coconuts for food. Piled them all up together and then went to sleep for the night.

"But when they were all asleep one man woke up, and he thought there might be a row about dividing the coconuts in the morning, so he decided to take his share. So he divided the coconuts into five piles. He had one coconut left over, and he gave that to the monkey, and he hid his pile and put the rest all back together.

"By and by the next man woke up and did the same thing. And he had one left over, and he gave it to the monkey. And all five of the men did the same thing, one after the other; each one taking a fifth of the coconuts in the pile when he woke up, and each one having one left over for the monkey. And in the morning they divided what coconuts were left, and they came out in five equal shares, again with one left over for the monkey. Of course each one must have known there were coconuts missing; but each one was guilty as the others, so they didn't say anything. How many coconuts were there in the beginning?"

This is an example of a diophantine problem, named after Diophantus of Alexandria, a Greek algebraist who was the first to extensively analyze equations with solutions in rational numbers. Now some diophantine equations have only one solution, some have no solutions, and others have more than one but finitely many solutions. But our problem here gives rise to a diophantine equation with infinitely many solutions. More precisely, there are infinitely many possible answers for the number of coconuts. Our problem is to find the smallest such positive number.

Let n denote the number of coconuts in the pile. The first man A divides n coconuts into 5 piles and has one left. Hence there is an integer a such that

$$n = 5a + 1. \tag{1}$$

After he has hidden his share of a coconuts and given the extra one to the monkey, there are $4a$ coconuts left. The second man B divides $4a$ coconuts into 5 piles and has one left. Hence there is an integer b such that

$$4a = 5b + 1. \tag{2}$$

After he has hidden his share of b coconuts and given the extra one to the monkey, there are $4b$ coconuts left. The third man C divides $4b$ coconuts into 5 piles and has one left. Hence there is an integer c such that

$$4b = 5c + 1. \tag{3}$$

After he has hidden his share of c coconuts and given the extra one to the monkey, there are $4c$ coconuts left. The fourth man D divides $4c$ coconuts into 5 piles and has one left. Hence there is an integer d such that

$$4c = 5d + 1. \tag{4}$$

After he has hidden his share of d coconuts and given the extra one to the monkey, there are $4d$ coconuts left. The last man E divides $4d$ coconuts into 5 piles and has one left. Hence there is an integer e such that

$$4d = 5e + 1. \tag{5}$$

After he has hidden his share of e coconuts and given the extra one to the monkey, there are $4e$ coconuts left, the number they found in the morning. They now divide $4e$ coconuts into 5 piles and has one left. Hence there is an integer f such that

$$4e = 5f + 1. \tag{6}$$

Our task is to find the smallest positive integer n such that equations (1)–(6) are soluble in positive integers a, b, c, d, e and f.

We now do a bit of algebra to simplify this problem. If you do not wish to go through this somewhat tedious calculation, head straight for the equation (11) below.

Multiplying equations (1) and (2) by 4 and 5 respectively, we obtain

$$4n = 20a + 4 \qquad \text{and} \qquad 20a = 25b + 5.$$

Substituting the second of these into the first, we obtain

$$4n = 25b + 9. \tag{7}$$

Multiplying equations (7) and (3) by 4 and 25 respectively, we obtain

$$16n = 100b + 36 \qquad \text{and} \qquad 100b = 125c + 25.$$

Substituting the second of these into the first, we obtain

$$16n = 125c + 61. \tag{8}$$

Multiplying equations (8) and (4) by 4 and 125 respectively, we obtain

$$64n = 500c + 244 \qquad \text{and} \qquad 500c = 625d + 125.$$

Substituting the second of these into the first, we obtain

$$64n = 625d + 369. \tag{9}$$

Multiplying equations (9) and (5) by 4 and 625 respectively, we obtain

$$256n = 2500d + 1476 \qquad \text{and} \qquad 2500d = 3125e + 625.$$

Substituting the second of these into the first, we obtain

$$256n = 3125e + 2101. \tag{10}$$

Multiplying equations (10) and (6) by 4 and 3125 respectively, we obtain

$$1024n = 12500e + 8404 \qquad \text{and} \qquad 12500e = 15625f + 3125.$$

Substituting the second of these into the first, we obtain

$$1024n = 15625f + 11529. \tag{11}$$

Our task is to find the smallest positive integer n such that equation (11) is soluble with some positive integer f. Unfortunately this equation is too hard to solve by trial and error.

There is an uncanny but beautifully simple solution to this problem involving the concept of negative coconuts, attributed to the Cambridge physicist Dirac, who claimed that he obtained this from the Oxford mathematician Whitehead, who claimed that he learned this from someone else.

The reasoning is as follows. Given that the pile of coconuts is divided 6 times into 5 piles, it is then clear that $5 \times 5 \times 5 \times 5 \times 5 \times 5 = 15625$ coconuts can be added to any answer for the number of coconuts to give the next higher answer. Indeed, any integer multiple of 15625 can be added to or subtracted from any answer to obtain another answer. This of course gives us infinitely many negative answers. These all satisfy the equation (11). While there is no small positive integer n that gives rise to a solution to the equation (11), a little trial and error with negative values of n comes up with a solution $n = -4$, with $f = -1$. Adding 15625 to this negative solution leads to the smallest positive solution $n = 15621$.

Imagine the first man A waking up to find a pile of -4 coconuts. He tosses a positive coconut to the monkey, so there is now a pile of -5 coconuts. He hides his share of -1 coconut, and leaves a pile of -4 coconuts for the second man B to discover. The same thing then happens to B, C, D and E in succession, so there is now a pile of -4 coconuts in the morning. After tossing a positive coconut to the monkey, there is now a pile of -5 coconuts, so each of them has -1 coconut more!