

A Simple Diophantine Equation

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We consider the simple diophantine equation

$$(1) \quad ax + by = c,$$

where a , b and c are given fixed integers. The problem is to find integers x and y , if any, such that this equation is satisfied. We also relate the study of this equation to our earlier study of congruence equations.

We shall consider only the special case when the integers a and b are non-zero and have no common factors. Then for every choice of the integer c , it can be shown that the equation (1) has integer solutions x and y . Indeed, we shall show in this case that there are infinitely many pairs of integers x and y such that the equation (1) is satisfied, and we devise a way to find them all.

Suppose first of all that $x = x_0$ and $y = y_0$ give an integer solution of the equation (1). This means that

$$(2) \quad ax_0 + by_0 = c.$$

Let $x_1 = x_0 + b$ and $y_1 = y_0 - a$. Then

$$ax_1 + by_1 = a(x_0 + b) + b(y_0 - a) = ax_0 + ab + by_0 - ba = ax_0 + by_0 = c,$$

in view of (2). This means that $x = x_1$ and $y = y_1$ also give an integer solution of the equation (1). More generally, for every integer k , let

$$(3) \quad x_k = x_0 + bk \quad \text{and} \quad y_k = y_0 - ak.$$

Then

$$ax_k + by_k = a(x_0 + bk) + b(y_0 - ak) = ax_0 + abk + by_0 - bak = ax_0 + by_0 = c,$$

in view of (2). This means that $x = x_k$ and $y = y_k$ also give an integer solution of the equation (1).

To summarize, if we can find integers x_0 and y_0 such that $x = x_0$ and $y = y_0$ give an integer solution of the equation (1), then (3) give infinitely many more integer solutions of the equation (1). In fact, these are precisely all the solutions.

Theorem. *Suppose that a , b and c are integers such that a and b are non-zero and have no common factors. Then for the diophantine equation (1), there exist integers x_0 and y_0 such that $x = x_0$ and $y = y_0$ give an integer solution of the equation (1). Furthermore, the integer solutions of the equation (1) are given precisely by (3), where k is any integer, positive, negative or zero.*

The problem is to find such a pair of integers x_0 and y_0 . However, the technique is relatively simple, and is essentially based on trial and error.

Suppose, for simplicity, that the integer $b > 0$. Then

$$ax_0 + by_0 = c \quad \text{precisely when} \quad ax_0 - c = -by_0.$$

On the other hand, $ax_0 - c = -by_0$ for some integer y_0 precisely when $ax_0 - c$ is a multiple of b . We therefore need to determine which integer values of x ensure that $ax - c$ is a multiple of b . But this is exactly the same problem as studying the congruence equation

$$ax \equiv c \pmod{b}.$$

We have already learned how to solve a congruence equation such as this. We simply check which of the integers $x = 0, 1, 2, \dots, b-1$ satisfy the congruence equation. In other words, we only need to check which of the integers $x = 0, 1, 2, \dots, b-1$ ensure that $ax - c$ is a multiple of b .

Example. Consider the diophantine equation $5x + 8y = 1$. Here we have $a = 5$, $b = 8$ and $c = 1$. Note that

$$5x - 1 = -8y,$$

so that any integer solution $x = x_0$ and $y = y_0$ of the diophantine equation must come from those values of x such that $5x - 1$ is a multiple of 8. We have the following table:

x	0	1	2	3	4	5	6	7
$5x - 1$	-1	4	9	14	19	24	29	34

Note that when $x = 5$, we have $5x - 1 = 24$ which is a multiple of 8. Now take $x_0 = 5$. Then the equation $5x_0 + 8y_0 = 1$ is satisfied when $25 + 8y_0 = 1$; in other words, when $y_0 = -3$. We can now conclude that the integer solutions of the diophantine equation $5x + 8y = 1$ are given by

$$x = x_0 + bk = 5 + 8k \quad \text{and} \quad y = y_0 - ak = -3 - 5k,$$

where k is any integer, positive, negative or zero. For instance, the choice $k = 100$ leads to the solution $x = 805$ and $y = -503$.

Example. Consider the diophantine equation $7x - 5y = 3$. Here we have $a = 7$, $b = -5$ and $c = 3$. Note that

$$7x - 3 = 5y,$$

so that any integer solution $x = x_0$ and $y = y_0$ of the diophantine equation must come from those values of x such that $7x - 3$ is a multiple of 5. We have the following table:

x	0	1	2	3	4
$7x - 3$	-3	4	11	18	25

Note that when $x = 4$, we have $7x - 3 = 25$ which is a multiple of 5. Now take $x_0 = 4$. Then the equation $7x_0 - 5y_0 = 3$ is satisfied when $28 - 5y_0 = 3$; in other words, when $y_0 = 5$. We can now conclude that the integer solutions of the diophantine equation $7x - 5y = 3$ are given by

$$x = x_0 + bk = 4 - 5k \quad \text{and} \quad y = y_0 - ak = 5 - 7k,$$

where k is any integer, positive, negative or zero. For instance, the choice $k = 10$ leads to the solution $x = -46$ and $y = -65$.