The Fibonacci Sequence

William Chen

We consider the sequence

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, \ldots$$

of positive integers where, apart from the first two terms, every term is the sum of the previous two terms. The sequence was first studied by Leonardo de Pisa, whose nickname was Fibonacci.

We show that this sequence is related to geometry and the golden ratio, and also comment on natural occurrences of the numbers in this fascinating sequence.

If we let F_n denote the n-th term of this sequence, then we clearly have

$$F_n = F_{n-1} + F_{n-2} \quad \text{for every } n = 3, 4, 5, \ldots,$$

with $F_1 = F_2 = 1$. This is the recursive definition of the sequence. In principle, we can find the value of F_n for any positive integer n. For example, after a lot of calculation, we may arrive at the conclusion that

$$F_{100} = 354224848179261915075.$$

An alternative way of evaluating this sequence is the Binet formula

$$F_n = \frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}} \quad \text{for every } n = 1, 2, 3, \ldots, \tag{1}$$

first discovered by Euler. This gives F_n explicitly for every positive integer n, without reference to earlier terms. However, this formula is given in terms of three irrational numbers

$$a = \frac{1+\sqrt{5}}{2}, \qquad b = \frac{1-\sqrt{5}}{2} \qquad \text{and} \qquad c = \sqrt{5},$$

and raising irrational numbers to high powers is not a pleasant exercise. We note the abbreviation

$$F_n = \frac{a^n - b^n}{c} \quad \text{for every } n = 1, 2, 3, \ldots,$$

where a, b and c are given above.

The terms of the Fibonacci sequence can be found in nature in surprising and interesting ways:

- The number of petals in certain varieties of flowers can be 3 (lily, iris), 5 (buttercup, columbine), 8 (cosmo), 13 (yellow daisy, marigold), 21 (English daisy) or 34 (oxeye daisy).

- Bracts in a pine cone spiral in different directions in 8 and 13 rows. Scales in pineapples spiral in different directions in 8, 13 and 21 rows. Seeds in the centre of a sunflower spiral in different directions in 55 and 89 rows.

These relationships with the Fibonacci sequence, though, are not fully understood.

Let us return to the mathematical aspects of the Fibonacci sequence. We have already observed that finding the value of F_n is not a very pleasant problem, so let us concentrate on other aspects of the sequence.

We first ask how the irrational numbers

$$a = \frac{1+\sqrt{5}}{2} \qquad \text{and} \qquad b = \frac{1-\sqrt{5}}{2}$$

arise. These two numbers are "conjugates", in the sense that they are the two solutions of the quadratic equation

$$x^2 = x + 1.$$

Note that $a > 0$, $b < 0$, and $a + b = 1$. We have

$$a = \frac{1+\sqrt{5}}{2} = 1.61803\ldots \qquad \text{and} \qquad b = \frac{1-\sqrt{5}}{2} = -0.61803\ldots$$

as reasonable approximations. The number a is known as the golden ratio or golden section, and is often denoted by Φ, so that we write

$$\Phi = \frac{1+\sqrt{5}}{2}.$$

Note that

$$\begin{aligned}
\Phi^2 &= \Phi + 1, \\
\Phi^3 &= \Phi\Phi^2 = \Phi(\Phi + 1) = \Phi^2 + \Phi = \Phi + 1 + \Phi = 2\Phi + 1, \\
\Phi^4 &= \Phi^2\Phi^2 = \Phi^2(\Phi + 1) = \Phi^3 + \Phi^2 = 2\Phi + 1 + \Phi + 1 = 3\Phi + 2, \\
\Phi^5 &= \Phi^3\Phi^2 = \Phi^3(\Phi + 1) = \Phi^4 + \Phi^3 = 3\Phi + 2 + 2\Phi + 1 = 5\Phi + 3, \\
\Phi^6 &= \Phi^4\Phi^2 = \Phi^4(\Phi + 1) = \Phi^5 + \Phi^4 = 5\Phi + 3 + 3\Phi + 2 = 8\Phi + 5,
\end{aligned}$$

and so on. Note now that

$$\begin{aligned}
\Phi^2 &= F_2\Phi + F_1, \\
\Phi^3 &= F_3\Phi + F_2, \\
\Phi^4 &= F_4\Phi + F_3, \\
\Phi^5 &= F_5\Phi + F_4, \\
\Phi^6 &= F_6\Phi + F_5,
\end{aligned}$$

and in general, we have

$$\Phi^n = F_n\Phi + F_{n-1} \quad \text{for every } n = 2, 3, 4, \ldots. \tag{2}$$

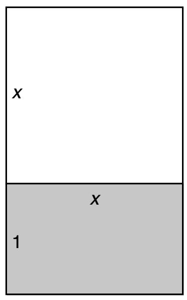
Note that formulas (1) and (2) have somewhat reverse roles. The Binet formula (1) gives F_n in terms of powers of the irrational numbers a and b, whereas the formula gives powers of Φ in terms of terms of the Fibonacci sequence.

It is also interesting to study the ratio of successive terms of the Fibonacci sequence. For every positive integer $n \geq 2$, we have evaluate the ratio F_n/F_{n-1}. The table below gives some useful data:

n	F_n	F_n/F_{n-1}	n	F_n	F_n/F_{n-1}
1	1		7	13	$13/8 = 1.625$
2	1	$1/1 = 1$	8	21	$21/13 = 1.61538\ldots$
3	2	$2/1 = 2$	9	34	$34/21 = 1.61904\ldots$
4	3	$3/2 = 1.5$	10	55	$55/34 = 1.61764\ldots$
5	5	$5/3 = 1.666\ldots$	11	89	$89/55 = 1.61818\ldots$
6	8	$8/5 = 1.6$	12	144	$144/89 = 1.61797\ldots$

Note that $F_n/F_{n-1} > \Phi$ when $n = 3, 5, 7, 9, 11$ and $F_n/F_{n-1} < \Phi$ when $n = 2, 4, 6, 8, 10, 12$. In fact, $F_n/F_{n-1} > \Phi$ for every odd integer $n \geq 3$ and $F_n/F_{n-1} < \Phi$ for every even positive integer. And the ratio F_n/F_{n-1} gets arbitrarily close to Φ if n is very large. Formally, we say that the sequence F_n/F_{n-1} has limit Φ as $n \to \infty$. This also means that for large positive integers n, we have a reasonably good approximation $F_n \approx \Phi F_{n-1}$.

We next study the relationship of the golden ratio Φ with geometry. Consider a rectangle of sides 1 and x. We now wish to attach a square of side x to this rectangle to form a bigger rectangle with sides x and $x + 1$. In the picture below, the original rectangle is shaded.



We now ask for which values of x will the larger rectangle be similar to the smaller rectangle.

We say that two rectangles are similar if the sides are proportional. In other words, we need

$$\frac{\text{short side of big rectangle}}{\text{long side of big rectangle}} = \frac{\text{short side of small rectangle}}{\text{long side of small rectangle}}.$$

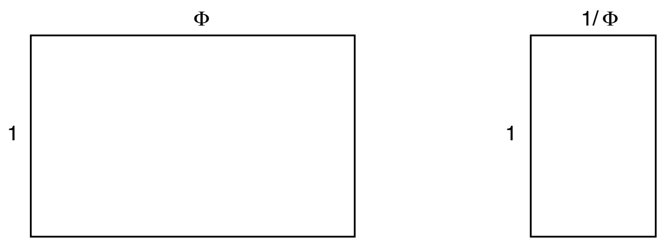
In our case, we need

$$\frac{x}{x+1} = \frac{1}{x}, \qquad \text{or} \qquad x^2 = x + 1.$$

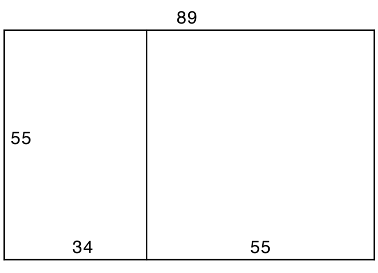
The only positive solution of this quadratic equation is

$$x = \Phi = \frac{1+\sqrt{5}}{2}.$$

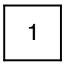
Such rectangles are called golden rectangles.



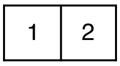
Rectangles with sides being successive terms of the Fibonacci sequence give good approximations to golden triangles.



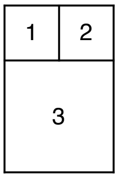
We conclude our discussion with a description of spiralling Fibonacci rectangles. We start with the first approximation to a golden rectangle, given by a rectangle of sides $F_1 = 1$ and $F_2 = 1$ which is labelled 1 in the pictures below. We then attach the square labelled 2 to this to obtain a rectangle of sides $F_2 = 1$ and $F_3 = 2$. We follow this by attaching the square labelled 3 to this to obtain a rectangle of sides $F_3 = 2$ and $F_4 = 3$.



first
generation

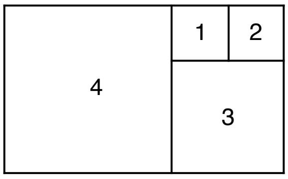


second
generation

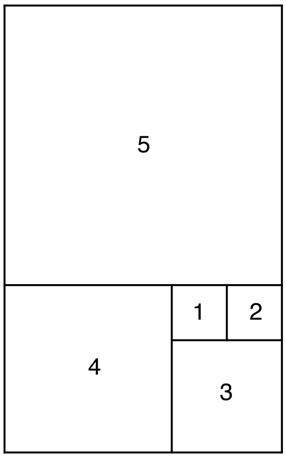


third
generation

We continue by attaching the square labelled 4 to obtain a rectangle of sides $F_4 = 3$ and $F_5 = 5$, and then attaching the square labelled 5 to obtain a rectangle of sides $F_5 = 5$ and $F_6 = 8$.

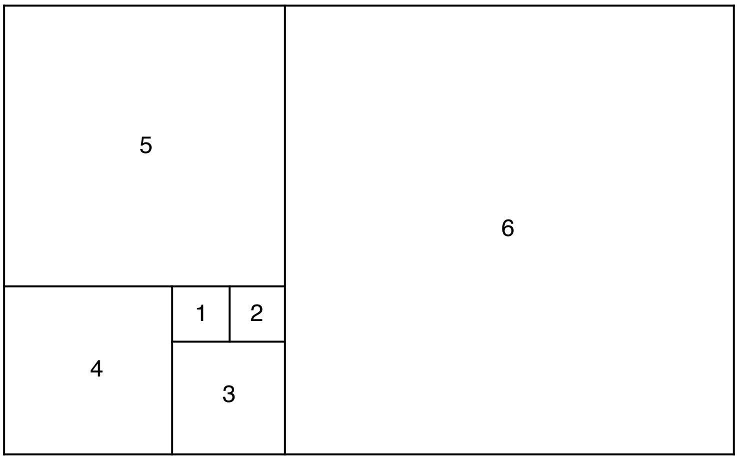


fourth
generation



fifth
generation

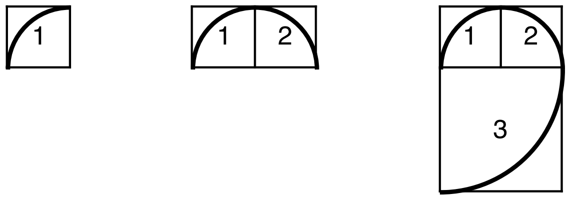
Then we continue by attaching the square labelled 6 to obtain a rectangle of sides $F_6 = 8$ and $F_7 = 13$.



sixth
generation

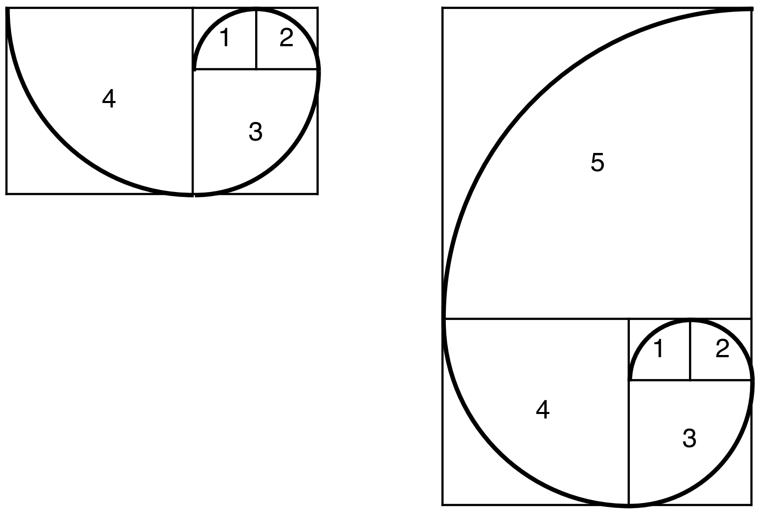
And so on.

We next add a circular arc to each square as shown. For the first three steps, we have the following.



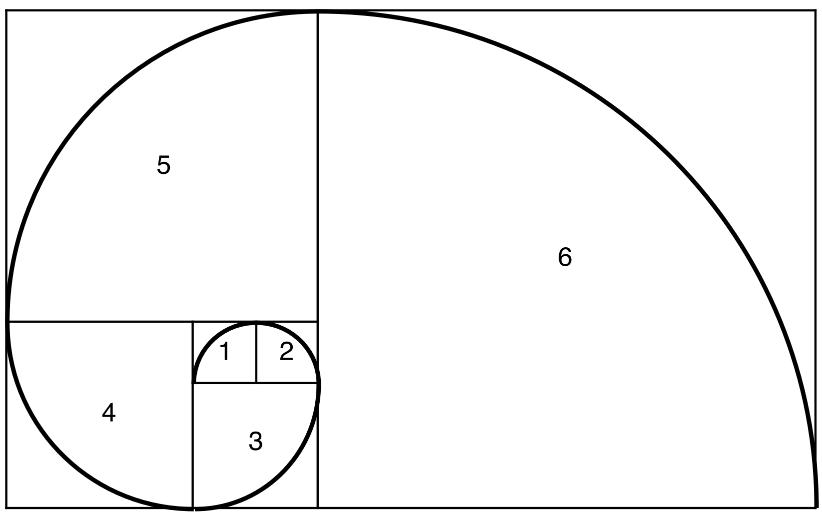
first generation second generation third generation

Note that we have been a little careful in placing each new square so that we actually get a smooth arc. Continuing in this way, the next two steps are as shown.



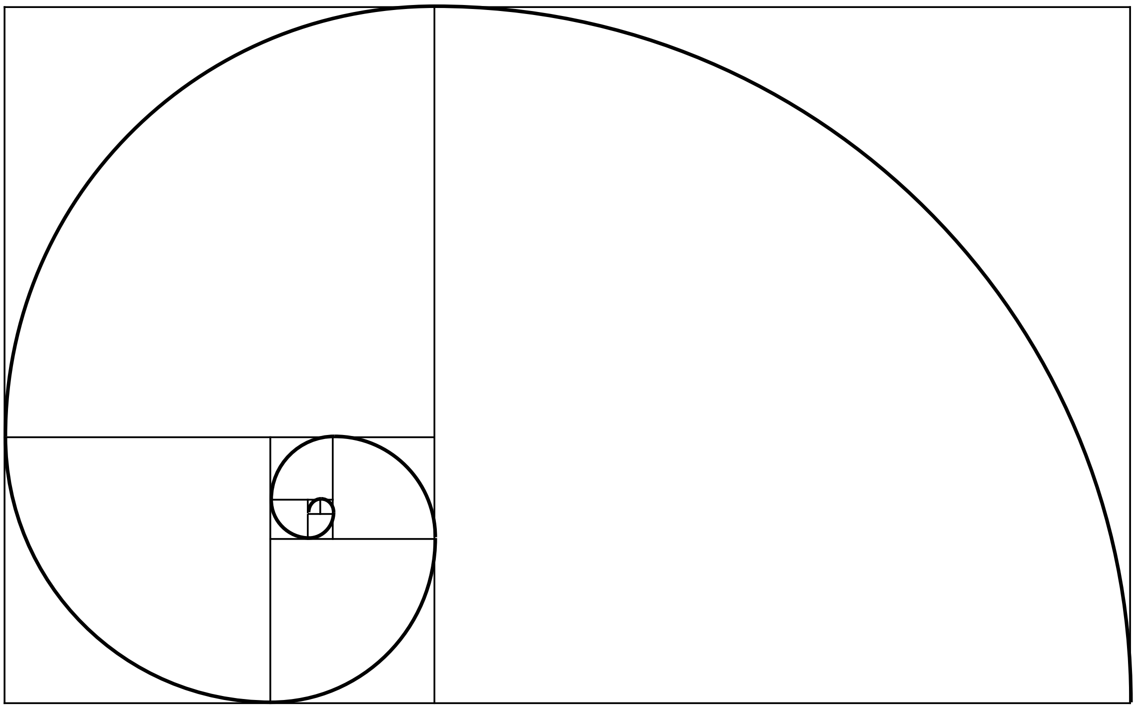
fourth generation fifth generation

The sixth step leads to the following.



sixth generation

The picture below shows the arc after ten steps.



The rectangle has sides $F_{10} = 55$ and $F_{11} = 89$.