

Fractal Geometry

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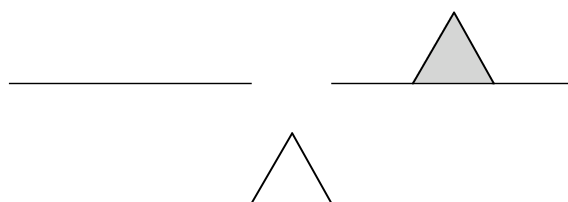
Over the last twenty or thirty years, a new type of geometry has enabled us to understand and even reconstruct some of nature's complex images. Here the two main rules are recursive replacement and self similarity. We give two examples of this new geometry of natural shapes, known in the literature as fractal geometry. Our two examples are the Koch snowflake and the Sierpinski gasket. We also look at the effect the two main rules have on perimeter and area.

To construct the Koch snowflake, we start first with an equilateral triangle. For simplicity, we may assume that the side length is equal to 1.

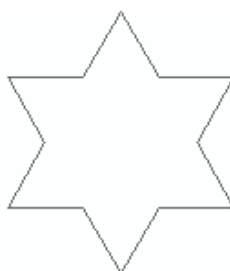


Note that each edge of the triangle is a line segment.

We now proceed to replace each of these line segments with a different object. Note that for each of these line segments, the inside of the triangle is on one side and the outside of the triangle is on the other. We now place an equilateral triangle on the outside of this line segment. This equilateral triangle has side length exactly one-third the length of the line segment, and its base coincides with the middle third of the line segment, as shown below in the picture on the right. We then replace the line segment with the new object shown at the bottom.

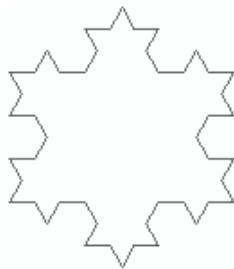


Note that this object consists of four line segments, each of which has length one-third the length of the line segment being replaced. If we make this replacement on each of the three edges of the original triangle, then we end up with the object shown below.



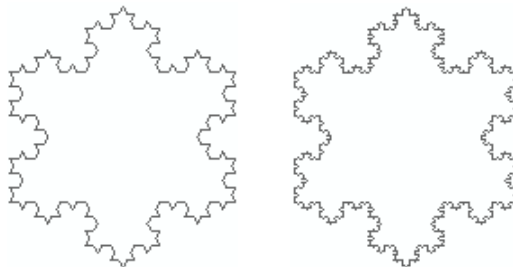
This first iterate of the original triangle has 12 edges, each of which is a line segment one-third the length of an edge of the original triangle.

We next proceed to replace each of these edges in the same way as before, except that there are more edges to replace and the scale is one-third as before. This is the notion of self symmetry. If we make this replacement on each of the edges of the first iterate of the original triangle, then we end up with the object shown below.



This second iterate of the original triangle has 48 edges, each of which is a line segment one-third the length of an edge of the first iterate of the original triangle, or one-ninth the length of an edge of the original triangle.

Proceeding in the same way, we obtain the third and fourth iterates of the original triangle, with 192 and 768 edges respectively.



Continuing in this way *ad infinitum* gives rise to the Koch snowflake.

One of the most interesting aspects of the Koch snowflake is its boundary. It is an extremely jagged curve, and usually known as the Koch curve. The main question is how long this curve is. Assume, as before, that we start with an equilateral triangle with side length equal to 1. Then the boundary of the original triangle has length

$$L_0 = 3.$$

To change from the original triangle to the first iterate, we remove the middle third of each edge and replace it with two edges each of length equal to the part that we have removed. It follows that each edge of the original triangle is replaced by edges with total length equal to four-thirds of the length of the original edge. Since we do the same to each edge of the original triangle, we conclude that the boundary of the first iterate has length

$$L_1 = \frac{4}{3} \times 3.$$

At the next iteration, each edge of the first iterate is replaced by edges with total length equal to four-thirds of the length of the edge. Hence we conclude that the boundary of the second iterate has length

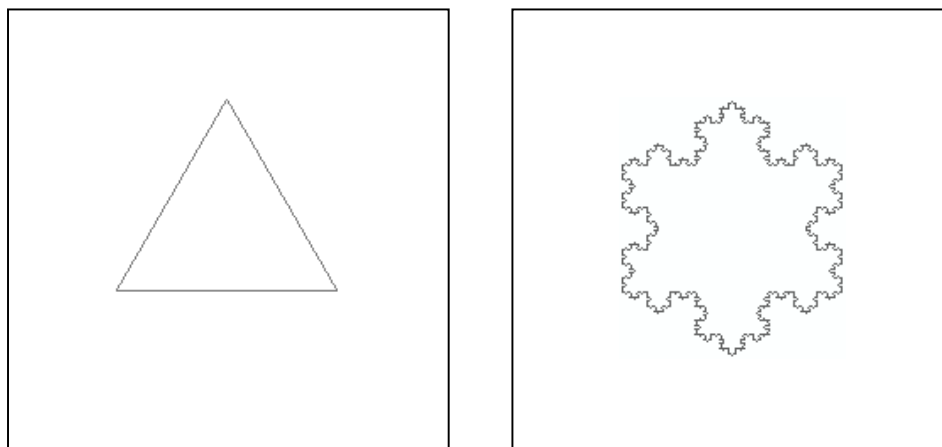
$$L_2 = \frac{4}{3} \times \frac{4}{3} \times 3.$$

Continuing in this way, we conclude that the boundary of the n -th iterate of the original triangle has length

$$L_n = \left(\frac{4}{3}\right)^n \times 3.$$

Clearly, this gets arbitrarily large if we allow n to be arbitrarily large. It follows that the Koch curve has infinite length.

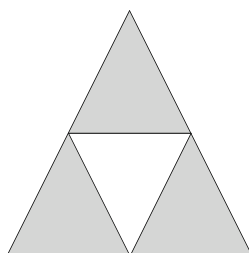
The total area of the Koch snowflake is clearly finite. To see this, note that if the original equilateral triangle has side length equal to 1, then it is not difficult to see that the Koch snowflake can be contained inside a square of side length 2, with total area 4.



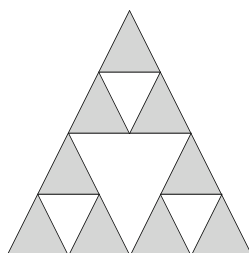
In fact, it can be shown that the area of the Koch snowflake is equal to 1.6 times the area of the original equilateral triangle.

The Koch snowflake is therefore an example of a shape with a finite area enclosed within an infinite boundary. This seems contrary to geometric intuition, but is characteristic of many shapes in nature. Less dramatic is the observation that if we take all the arteries, veins and capillaries in the human body, then they occupy a relative small fraction of the body. Yet if we were able to lay them out end to end, we would find that the total length would come to over 60000 kilometres.

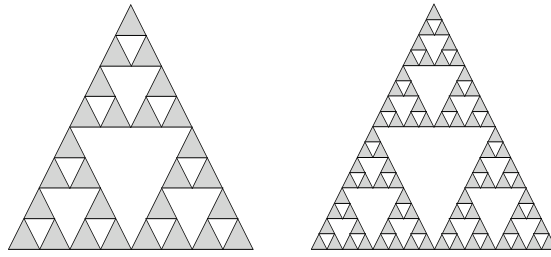
Our second example is the Sierpinski gasket. To construct this, we start with an arbitrary triangle. For simplicity, let us assume that this starting triangle has area 1. We split this triangle into four triangles by joining the midpoints of the three edges of the triangle. It is not difficult to convince ourselves that the four smaller triangles have equal areas. We then remove the middle triangle. In the picture below of this first iterate, the shaded parts represent the remaining parts of the original triangle after the middle small triangle has been removed. We adopt the convention that we never remove the edges of the middle triangle but only its interior.



For each of the three remaining smaller triangles, we split the triangle into four yet smaller triangles and then remove the middle triangle. This is where self symmetry comes in. If we pretend that we are only seeing one of these three smaller triangles at a time, then we are doing to it what we did to the original triangle, except that the scale has changed. We now have the second iterate of the original triangle shown below.



Proceeding in the same way, we obtain the third and fourth iterates of the original triangle.



Note that the picture on the right has three copies of the picture on the left but in smaller size. Continuing in this way *ad infinitum* gives rise to the Sierpinski gasket.

Next, we wish to find out the area of the Sierpinski gasket. Assume, as before, that we start with a triangle of area

$$A_0 = 1.$$

At the first iteration, we remove one of the four smaller triangles of equal area and keep the remaining three triangles, so that the total area of the first iterate is equal to

$$A_1 = \frac{3}{4} \times 1.$$

At the second iteration, we remove one quarter of each of the triangles and keep the remaining three quarters, so that the total area of the second iterate is equal to

$$A_2 = \frac{3}{4} \times \frac{3}{4} \times 1.$$

Continuing in this way, we conclude that the total area of the n -th iterate of the original triangle is equal to

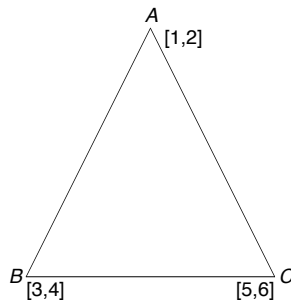
$$A_n = \left(\frac{3}{4}\right)^n \times 1.$$

Clearly, this gets arbitrarily small if we allow n to be arbitrarily large. It follows that the Sierpinski gasket has zero area.

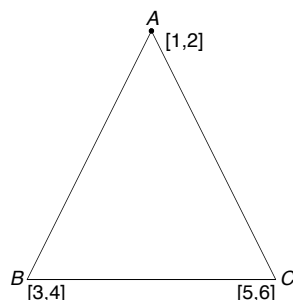
We can also consider the boundary of the Sierpinski gasket. The length of the boundary of the n -th iterate of the original triangle is the total length of the boundaries of all the shaded small triangles in the n -th iterate. It can be shown that this gets arbitrarily large as n gets arbitrarily large. It follows that the Sierpinski gasket has infinite boundary.

The Sierpinski gasket is therefore an example of a shape with zero area enclosed within an infinite boundary. Sierpinski gaskets serve as very versatile and efficient antennas for mobile phones, wireless modems and GPS receivers.

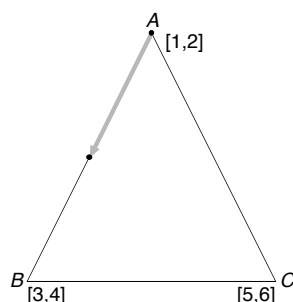
We conclude this discussion by looking at a game of chance. We start with an arbitrary triangle with vertices A , B and C , as well as a fair die. For each of the vertices of the triangle, we assign two of the values of the die in such a way that all six values from 1 to 6 are used. In our illustration here, we have assigned the values 1 and 2 to the vertex A , the values 3 and 4 to the vertex B , and the values 5 and 6 to the vertex C .



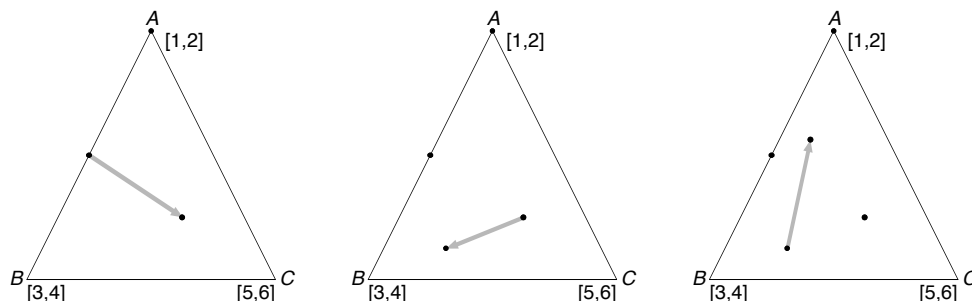
At the start of the game, we toss the die, and place a point on the winning vertex. For example, if we roll the number 1 or 2, then we place a point at the vertex A .



We toss the die again, and then place a second point between the first point and the second winning vertex. For example, if we roll the number 3 or 4, then we place a second point halfway between vertex A , where the first point is, and the winning vertex B .



At every subsequent toss, we place a point halfway between the previous point and the winning vertex. For example, if our next three tosses come up with 6, 4 and 1, then we place points as shown below.



We keep on playing this game. After perhaps 50 or 100 tosses, we shall have a triangle dotted with 50 or 100 points. However, if we repeat this play thousands of times, then the positions of the thousands of points that we put down will give us a picture which is visually a reasonably good approximation of the Sierpinski gasket. The longer we play this game, the better the approximation becomes!

Note that the positions of the points are subject to chance, so it is somewhat surprising that we end up with a highly predictable pattern.