Regular Polygons

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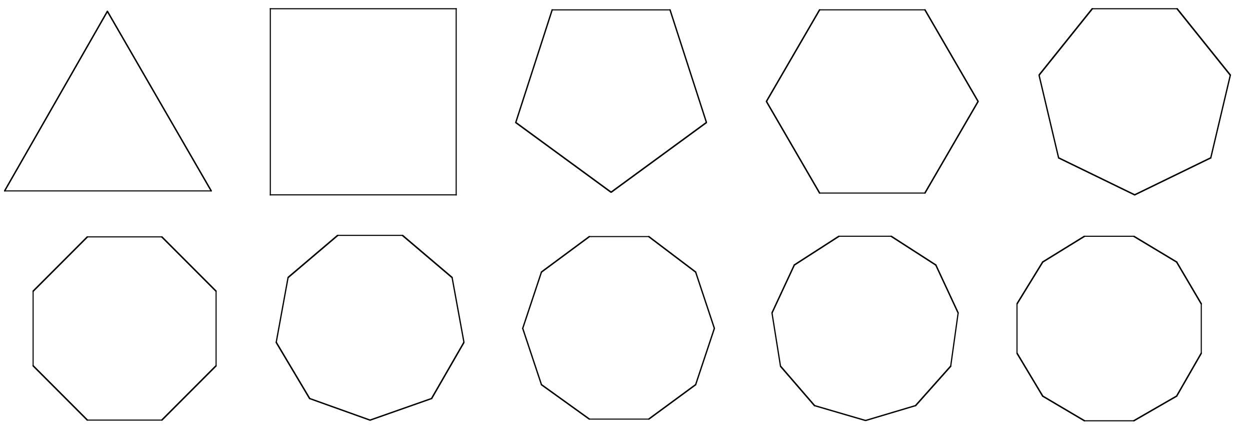
We make a brief study of regular polygons, construct a few simple ones by ruler and compasses, and look at their symmetry properties, both mirror and rotational.

A polygon, literally "many-angle", is a closed planar path composed of a finite number of line segments, called sides or edges. The points where the sides meet are called vertices. If a polygon is simple, then its sides and vertices give rise to the boundary of a polygonal region, and the term polygon sometimes also describes the interior of the polygonal region. It is easy to see that the number of vertices of a polygon is equal to the number of sides. A polygon is said to be regular if all the edges are of the same length and all the internal angles are equal.

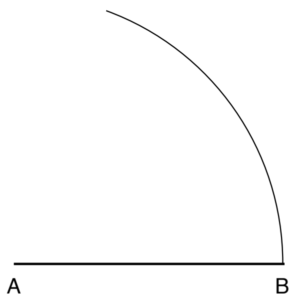
Polygons are named according to the number of sides, with a Greek-derived numerical prefix and with the suffix "gon", such as pentagon (5 sides) and hexagon (6 sides). Occasionally we use the term k-gon to denote a polygon with k sides. Hence a 3-gon is a triangle.

Below we find pictures of regular polygons of 3 to 12 sides, known respectively as the regular trigon, tetragon, pentagon, hexagon, heptagon, octagon, enneagon, decagon, hendecagon and dodecagon. Of course, a regular trigon is simply an equilateral triangle and a regular tetragon is simply a square. An enneagon is also sometimes known as a nonagon.

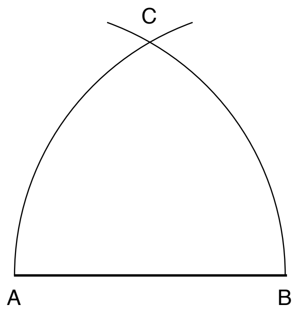


Some regular polygons can be constructed by ruler and compasses. Here we shall restrict our discussion to three very simple cases, the equilateral triangle, the square and the regular hexagon.

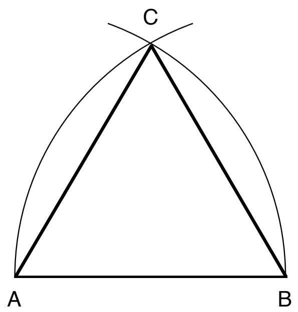
To construct an equilateral triangle, we start with one edge, denoted by AB. We then use a pair of compasses and draw a circular arc centred at the point A and passing through the point B as shown below.



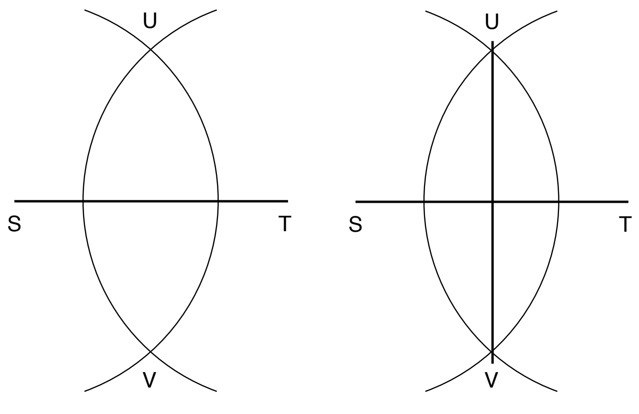
Without altering the compasses, we then draw another circular arc centred at the point B and passing through the point A as shown below. The two arcs intersect at a point C.



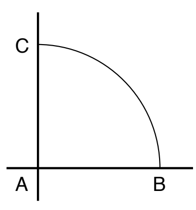
Now join the point C to the points A and B to obtain an equilateral triangle.



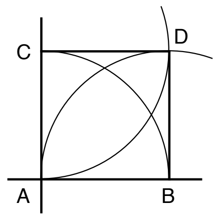
We next turn our attention to constructing a square. The first step is to construct two lines that are perpendicular to each other. We start with one line segment ST. Using a pair of compasses, we draw a circular arc centred at the point S. Without altering the compasses, we draw a circular arc centred at the point T. The two arcs now intersect at the points U and V as shown below on the left. We now simply draw a straight line to pass through the points U and V as shown below on the right.



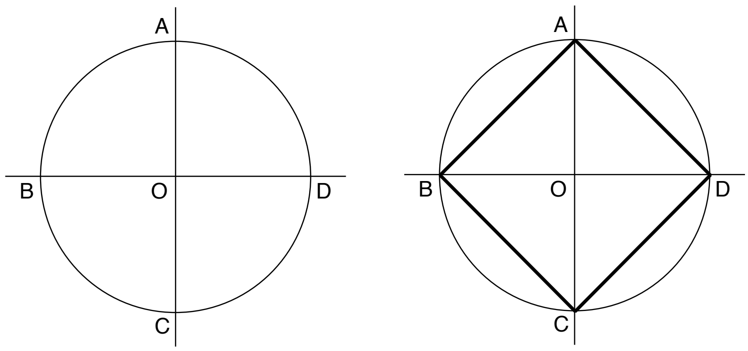
The line segment UV is perpendicular to the line segment ST. Now that we have two perpendicular lines, we can proceed to construct a square. Here we describe two techniques, the first of which is slightly more complicated. Let AB be one side of our square. Using a pair of compasses, we draw a circular arc centred at the point A and passing through the point B. The arc intersects the perpendicular line at the point C as shown below.



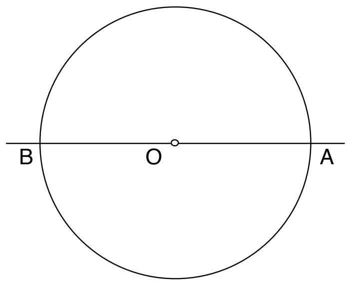
Clearly AC is perpendicular to AB and of the same length. We need to construct the two other sides of the square. Without altering the compasses from before, we draw a circular arc centred at the point B and passing through the point A, and another circular arc centred at the point C and passing through the point A. These two arcs now intersect at a point D as shown below.



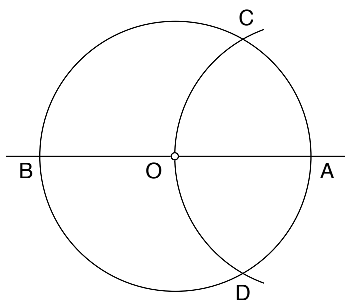
We now simply join the point D to the points B and C to obtain a square ABDC. The second technique is quite a lot simpler. We start with two perpendicular lines intersecting at the point O. Using a pair of compasses, we now draw a circle centred at the point O. This circle intersects the two perpendicular lines at the points A, B, C and D as shown below on the left. The line segments AB, BC, CD and AD form the four sides of a square as shown below on the right.



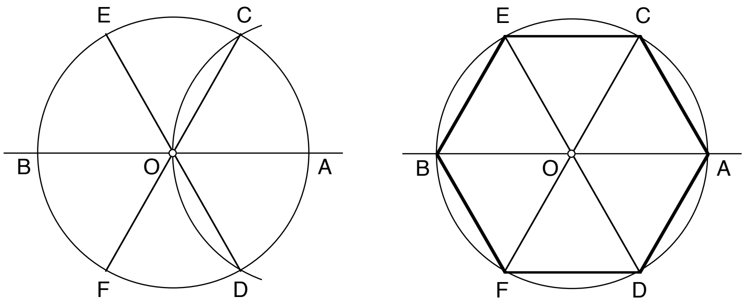
To construct a regular hexagon, we start with a line and pick a point O. Using a pair of compasses, we draw a circle centred at the point O. This circle intersects the line at the points A and B as shown below.



We next draw a circular arc centred at the point A and passing through the point O. This arc intersects the circle at the points C and D as shown below.



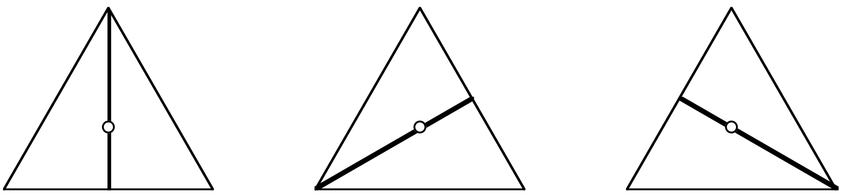
We then draw a straight line through the points O and C, and a straight line through the points O and D. These intersect the circle at the points E and F as shown below on the left.



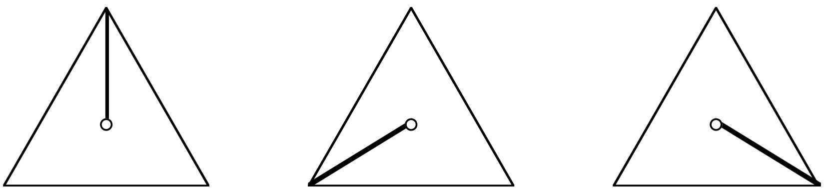
We now join the pairs of adjacent points on the circle to obtain a regular hexagon ACEBFD as shown above on the right.

We conclude our discussion here by studying symmetry properties of regular polygons.

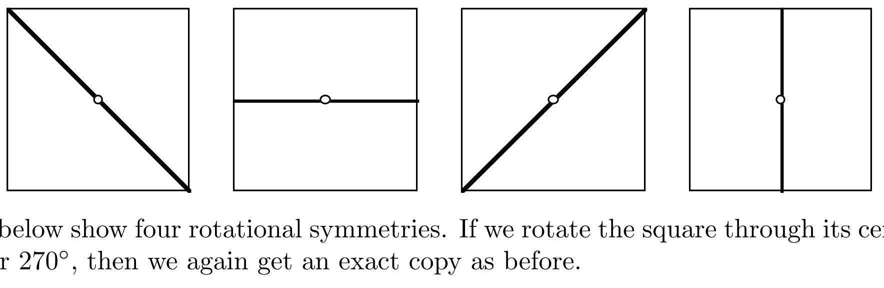
We begin by looking at the equilateral triangle. The three pictures below show three axes of reflection symmetry. If we reflect the triangle across an axis indicated, then we get an exact copy as before. Each of these axes of symmetry joins a vertex to the midpoint of the opposite side.



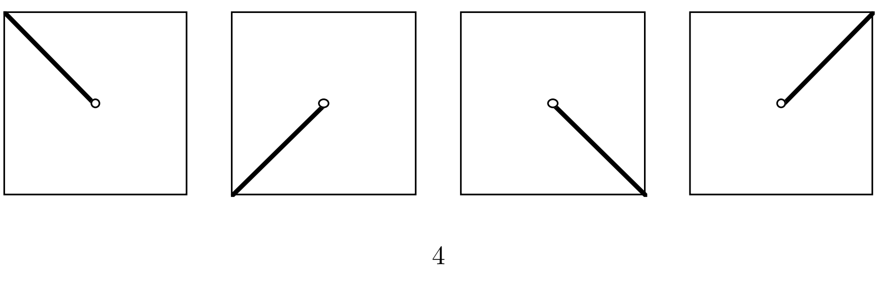
The three pictures below show three rotational symmetries. If we rotate the triangle through its centre anticlockwise by 0°, 120° or 240°, then we again get an exact copy as before.



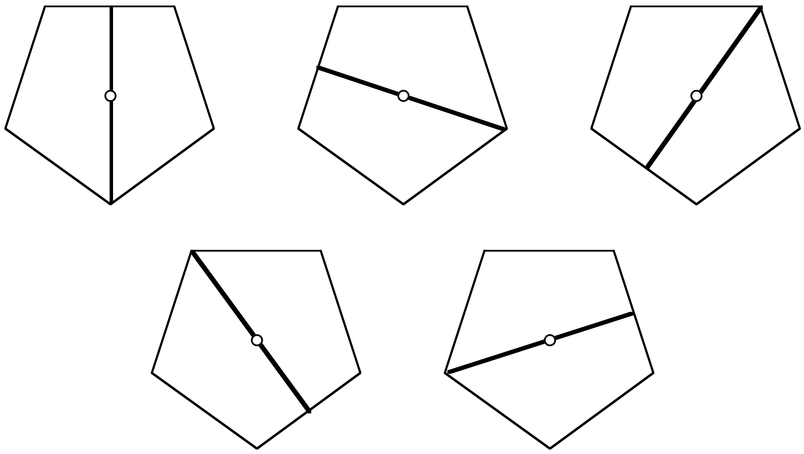
Next we look at the square. The four pictures below show four axes of reflection symmetry. If we reflect the square across an axis indicated, then we get an exact copy as before. Each of these axes of symmetry either joins two opposite vertices or joins the midpoints of two opposite sides.



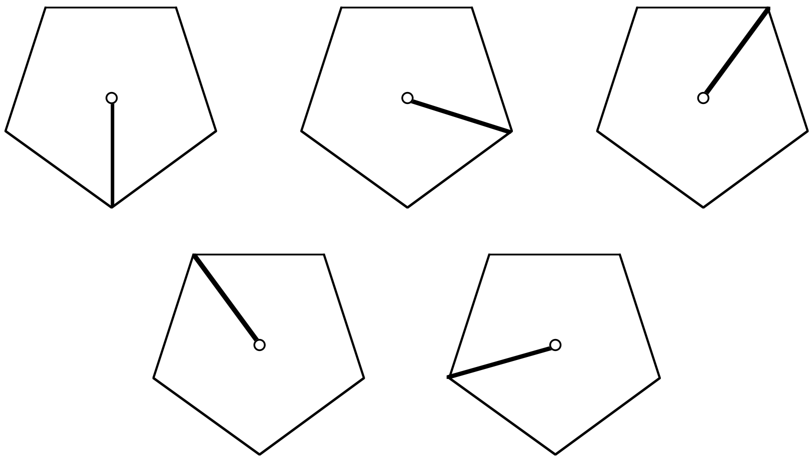
The four pictures below show four rotational symmetries. If we rotate the square through its centre anticlockwise by 0°, 90°, 180° or 270°, then we again get an exact copy as before.



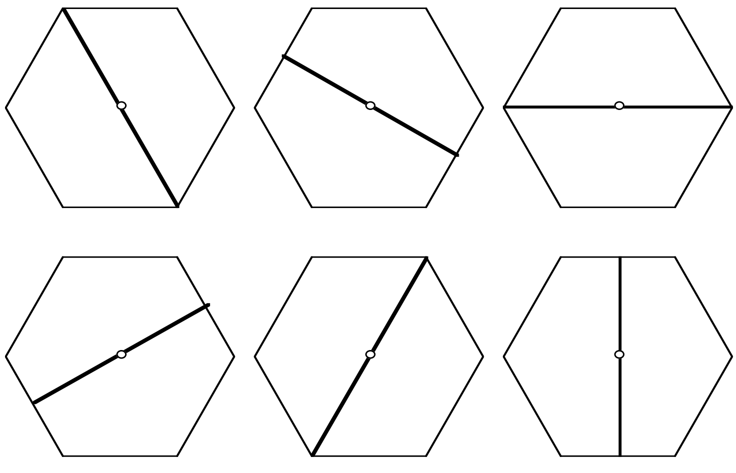
Next we look at the regular pentagon. The five pictures below show five axes of reflection symmetry. If we reflect the pentagon across an axis indicated, then we get an exact copy as before. Each of these axes of symmetry joins a vertex to the midpoint of the opposite side.



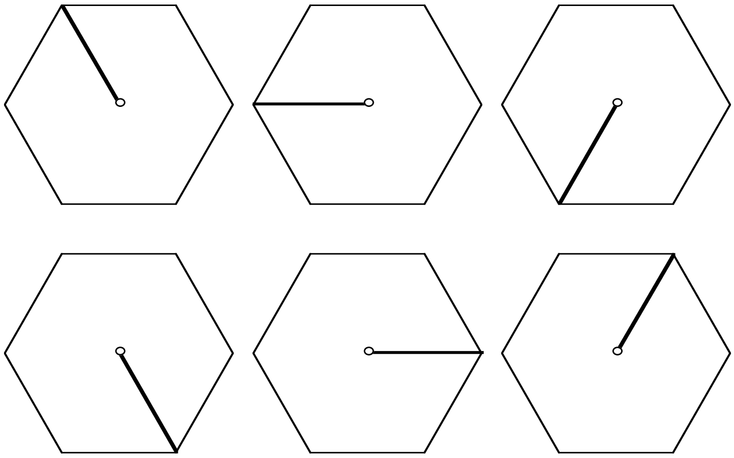
The five pictures below show five rotational symmetries. If we rotate the pentagon through its centre anticlockwise by 0°, 72°, 144°, 216° or 288°, then we again get an exact copy as before.



Finally, we look at the regular hexagon. The six pictures below show six axes of reflection symmetry. If we reflect the hexagon across an axis indicated, then we get an exact copy as before. Each of these axes of symmetry either joins two opposite vertices or joins the midpoints of two opposite sides.



The six pictures below show six rotational symmetries. If we rotate the hexagon through its centre anticlockwise by $0°$, $60°$, $120°$, $180°$, $240°$ or $300°$, then we again get an exact copy as before.



A little thought leads to the following conclusion for a regular k-gon.

- There are k axes of reflection symmetry. If k is odd, then each of these axes of symmetry joins a vertex to the midpoint of the opposite side. If k is even, then each of these axes of symmetry either joins two opposite vertices or joins the midpoints of two opposite sides.

- There are k rotational symmetries, corresponding to anticlockwise rotation of $0°$, $d°$, $2d°$, $\ldots$, $(k-1)d°$, where $d = 360/k$.