Platonic Solids

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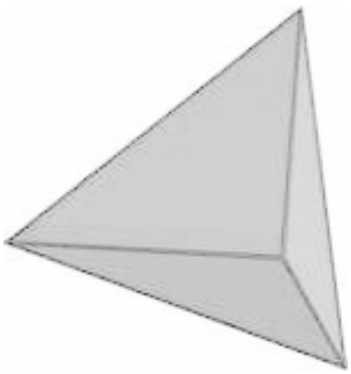
In two dimensions, we have polygons. It is easy to see that regular k-gons exist for any integer $k \geq 3$. In three dimensions, the analogues of polygons are polyhedrons. We may imagine that there are infinitely many different regular polyhedrons. In fact, there are only five. We shall see what they are, and explain why there are no more.

A regular polyhedron is a solid whose faces are regular polygons of the same shape and size, and with identical interior angles at its vertices. A simple example is the cube, where all the faces are squares of the same size, and where each vertex meets three faces.

There are only five regular polyhedrons, namely the tetrahedron, hexahedron (cube), octahedron, dodecahedron and icosahedron, with 4, 6, 8, 12 and 20 faces respectively.





The following table summarizes some of their properties:

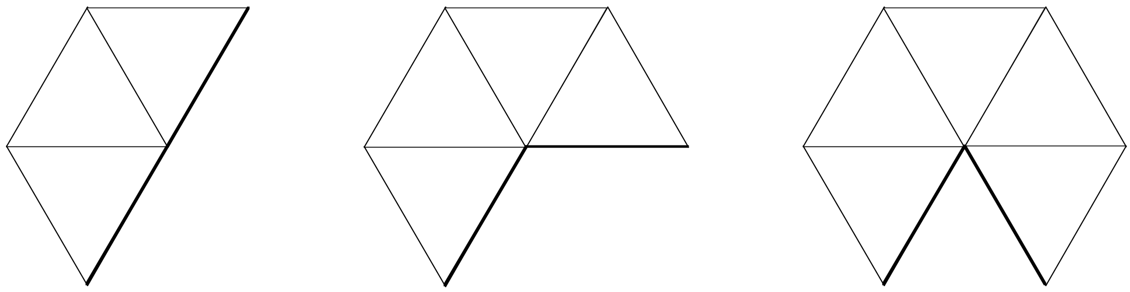
polyhedron	number of faces F	shape of faces	number of vertices V	number of edges E	number of edges at each vertex
tetrahedron	4	3-gon	4	6	3
hexahedron	6	4-gon	8	12	3
octahedron	8	3-gon	6	12	4
dodecahedron	12	5-gon	20	30	3
icosahedron	20	3-gon	12	30	5

These five regular polyhedrons are known as the Platonic solids. They were known to the ancient Greeks, and were described by Plato in his volume *Timaeus* around 350 BC. Indeed, Plato equated the tetrahedron, the hexahedron, the octahedron and the icosahedron with the four "elements" fire, earth, air and water respectively, and the dodecahedron with the stuff of which the constellations and heavens were made. A thousand years earlier, the neolithic people of Scotland developed the five solids, and stone models they made are kept in the Ashmolean Museum in Oxford.

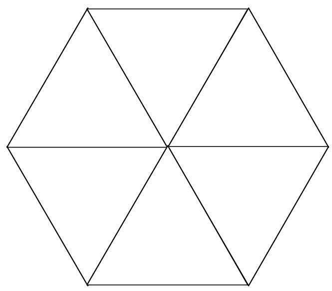
To prove that these five regular solids exist is beyond the scope of our present discussion. However, we shall first of all indicate why they are plausible. In doing so, we shall also understand why these are the only possibilities. We shall also indicate briefly how some of these can be constructed.

Clearly each vertex of a polyhedron must have at least three faces. For a regular polyhedron, these faces must be regular polygons – equilateral triangles, squares, regular pentagons, and so on.

Consider first of all the case where the face is an equilateral triangle. Since each vertex must meet at least three faces, the possibilities may be to put three, four or five such faces together at a given vertex as shown, and fold at the common edges sufficiently to join the edges indicated in bold.

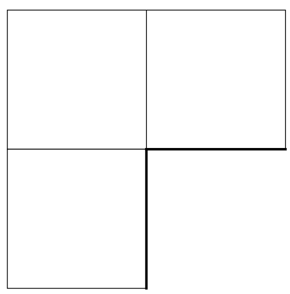


These give rise to the tetrahedron, octahedron and icosahedron respectively. On the other hand, if we put six such faces together at a given vertex, then it is clear that we cannot fold to make a three dimensional object.

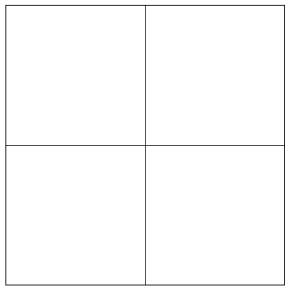


Hence the tetrahedron, octahedron and icosahedron are the only regular polyhedrons with triangular faces.

Consider next the case where the face is a square. Since each vertex must meet at least three faces, the possibility may be to put three such faces together at a given vertex as shown, and fold at the common edges sufficiently to join the edges indicated in bold.

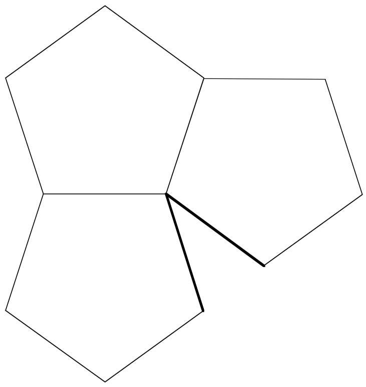


This gives rise to the hexahedron (cube). On the other hand, if we put four such faces together at a given vertex, then it is clear that we cannot fold to make a three dimensional object.



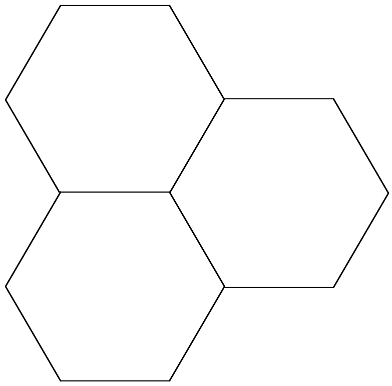
Hence the hexahedron (cube) is the only regular polyhedron with square faces.

Consider next the case where the face is a regular pentagon. Since each vertex must meet at least three faces, the possibility may be to put three such faces together at a given vertex as shown, and fold at the common edges sufficiently to join the edges indicated in bold.



This gives rise to the dodecahedron. On the other hand, it is clear that no vertex can accommodate four regular pentagons on the plane. Hence the dodecahedron is the only regular polyhedron with pentagonal faces.

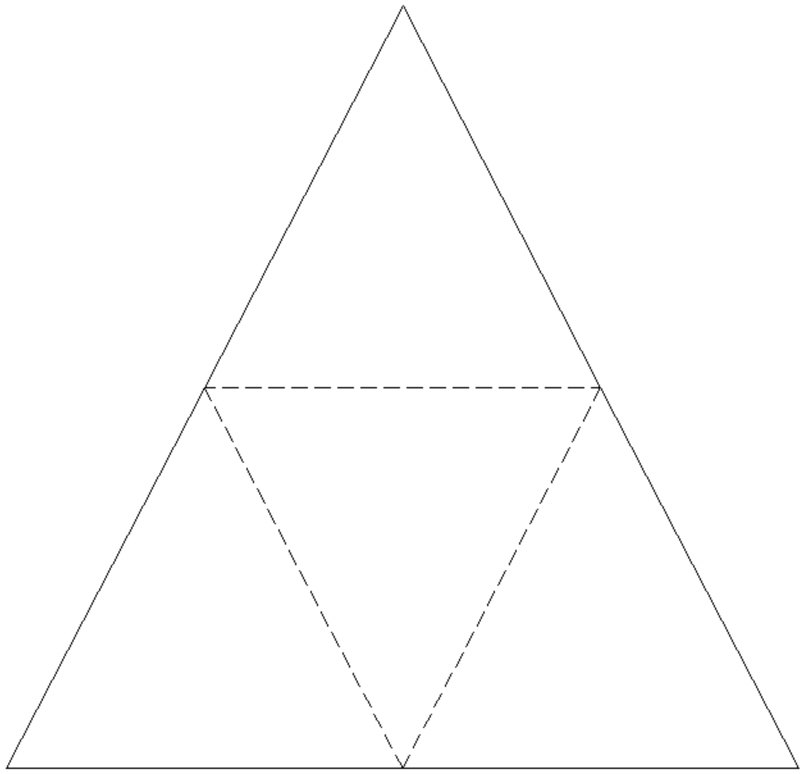
Consider next the case where the face is a regular hexagon. Since each vertex must meet at least three faces, the possibility may be to put three such faces together at a given vertex, and it is clear that we cannot fold to make a three dimensional object.



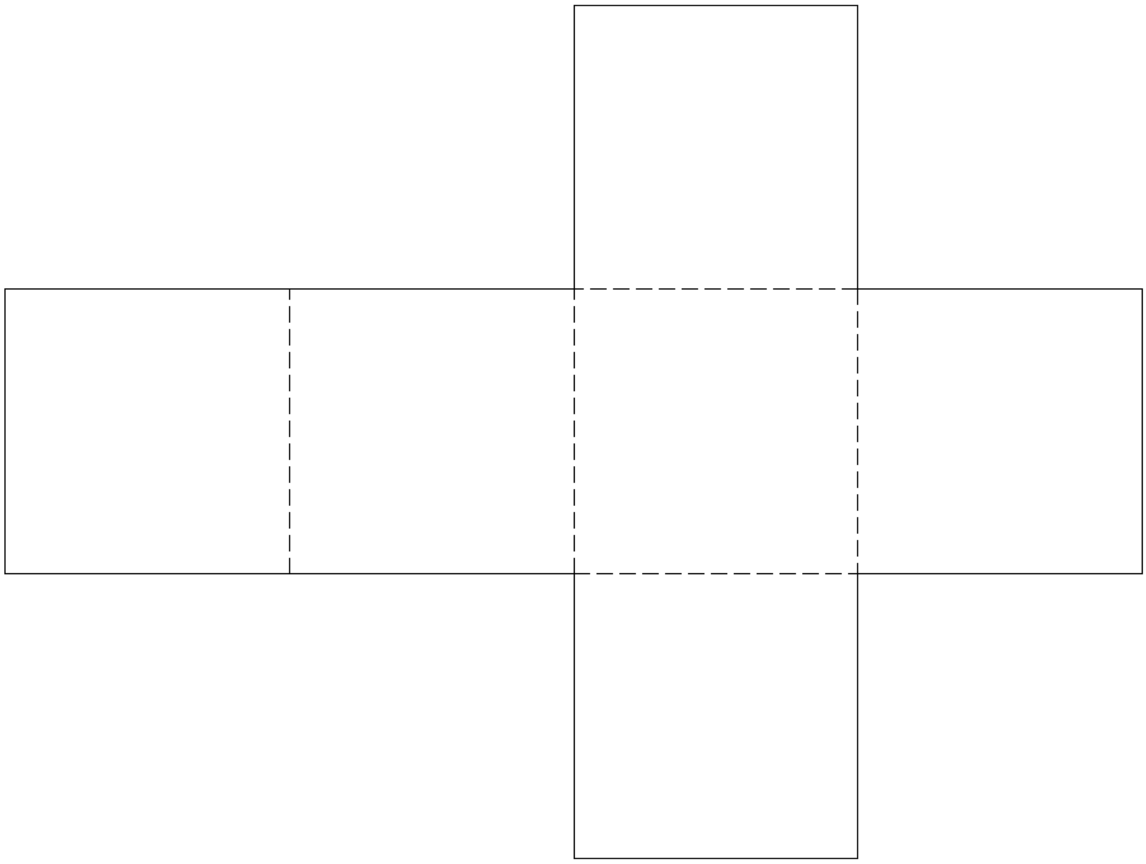
Hence there is no regular polyhedron with hexagonal faces. Finally, we note that no vertex can accommodate three regular k-gons on the plane when $k \geq 7$. Hence there is no regular polyhedron with k-gonal faces when $k \geq 7$.

The Platonic solids can be constructed by using nets of equilateral triangles, squares and regular pentagons. We shall now include these nets. Gently fold at the broken lines and use a little sellotape.

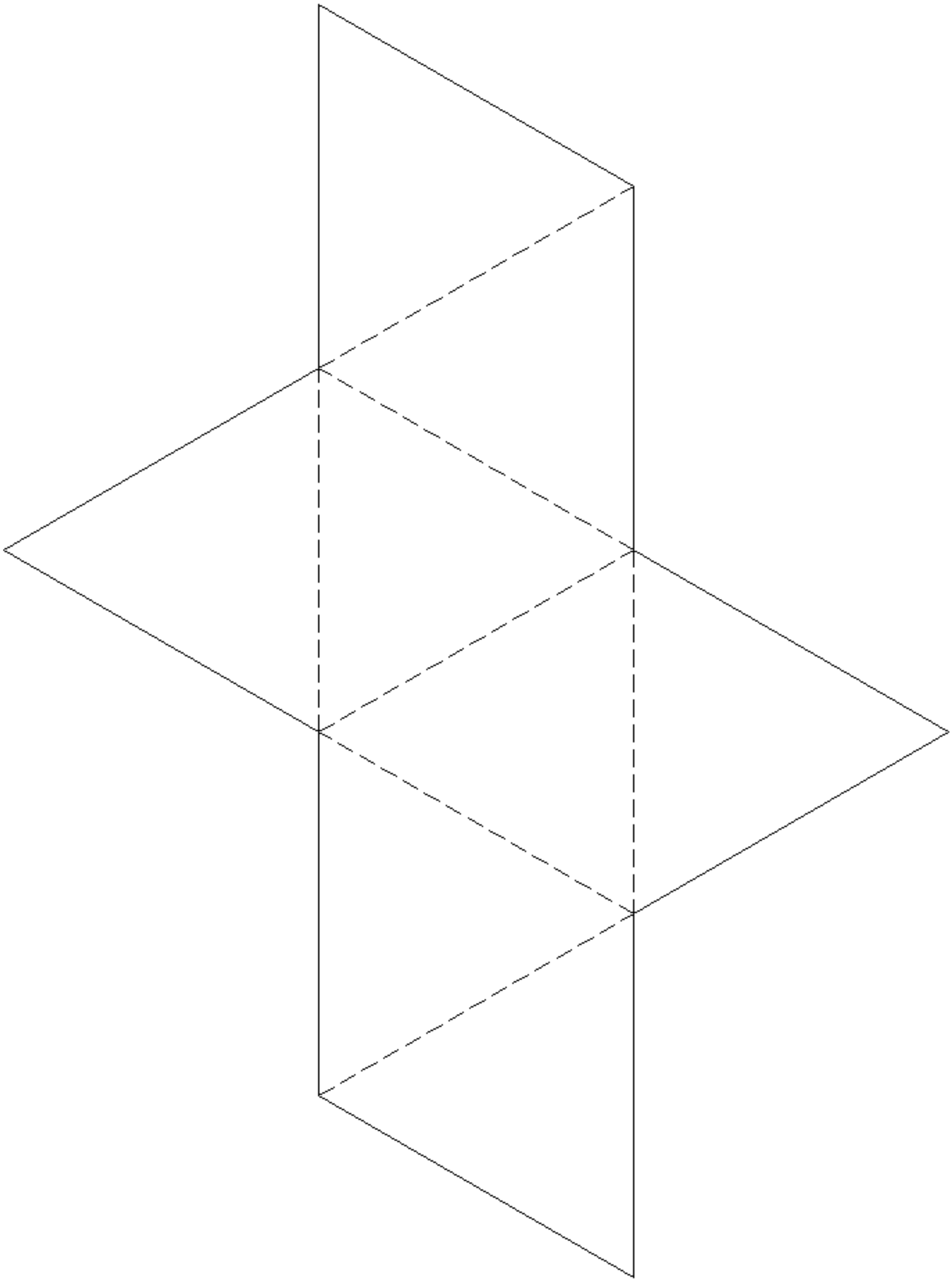
To construct the regular tetrahedron, we can use the following:



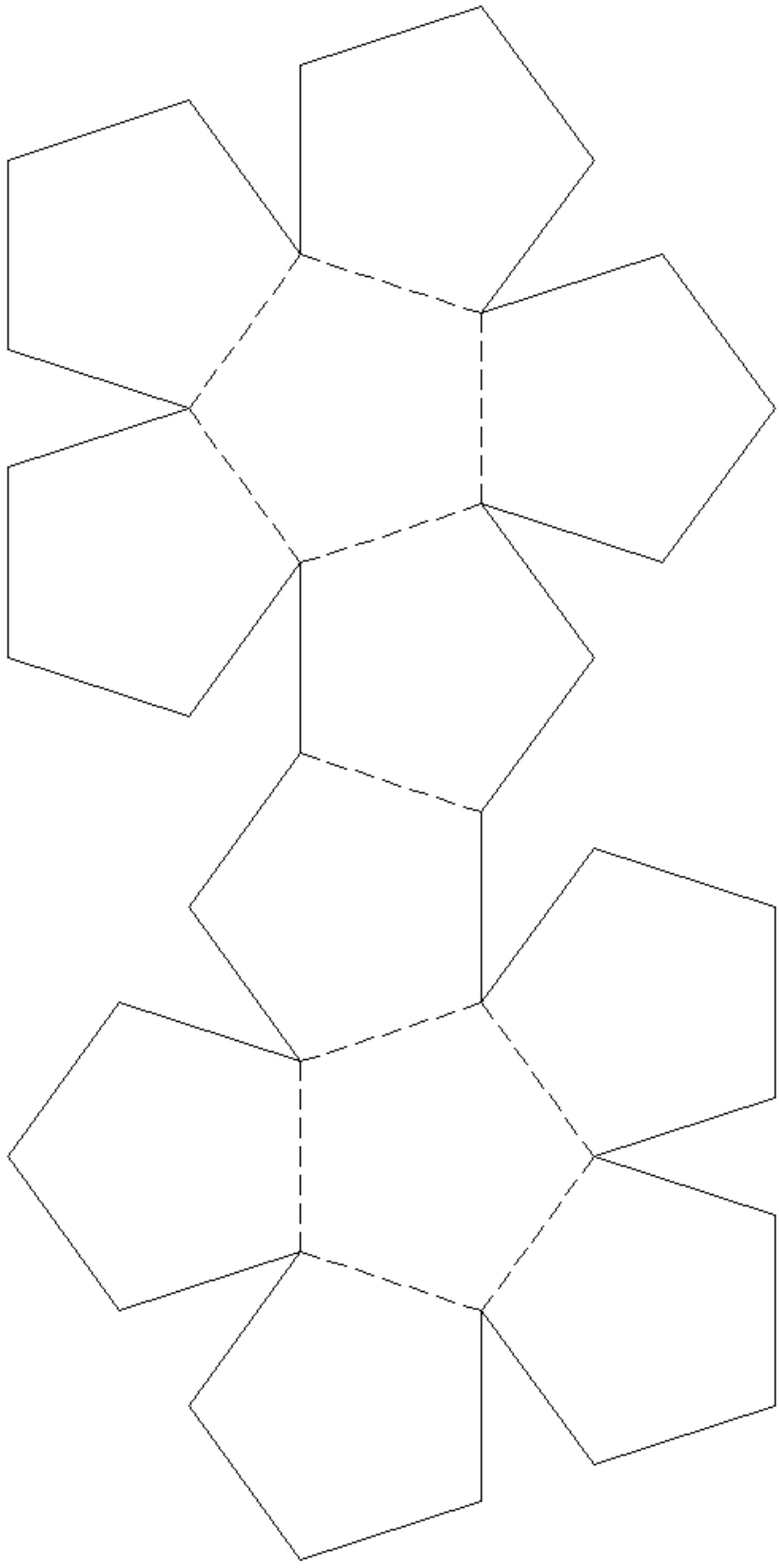
To construct the regular hexahedron (cube), we can use the following:



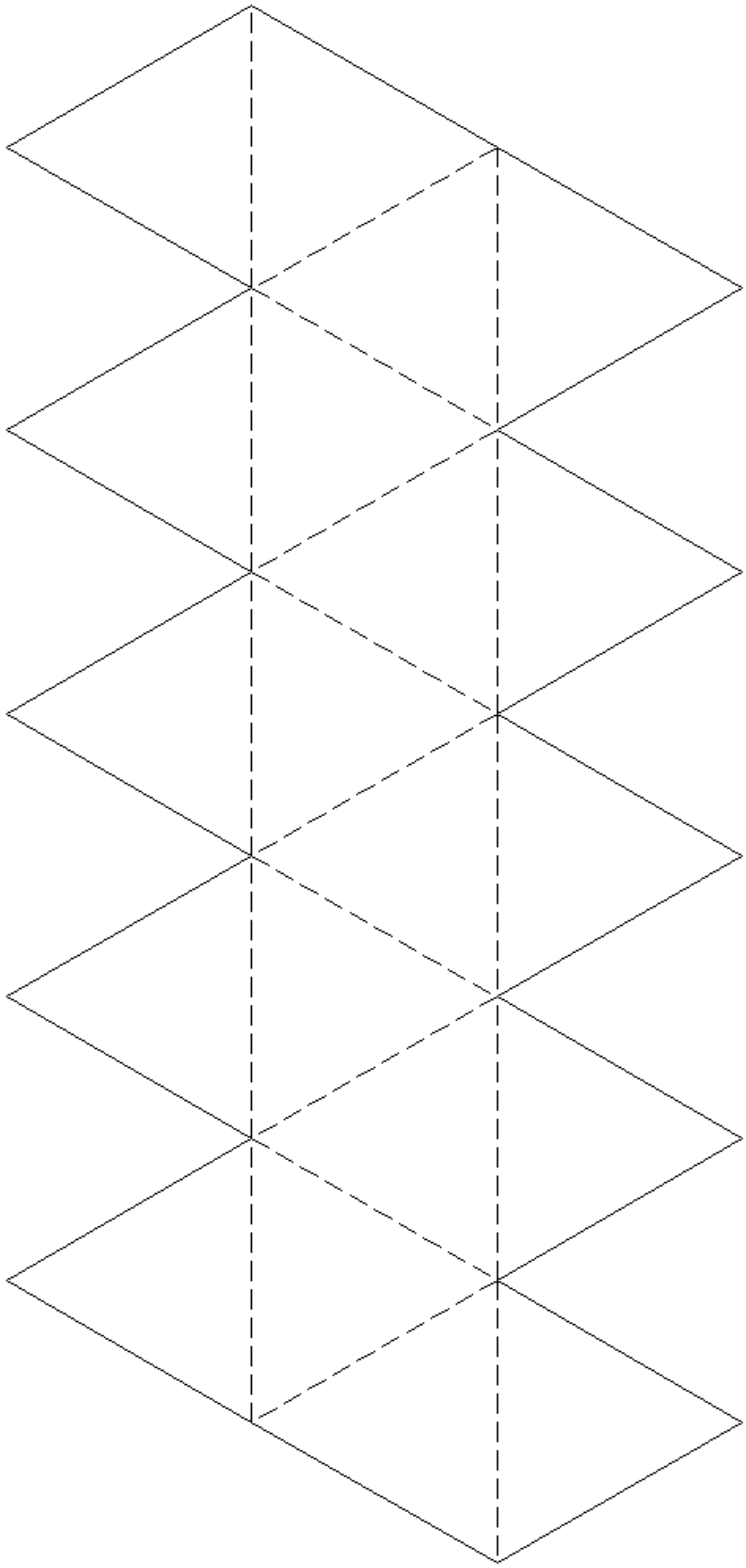
To construct the regular octahedron, we can use the following:



To construct the regular dodecahedron, we can use the following:



To construct the regular icosahedron, we can use the following:



Another way to understand the five Platonic solids is Euler's formula, that for any given polyhedron, we have

$$F + V - E = 2,$$

where F denotes the number of faces, E denotes the number of edges and V denotes the number of vertices. If we check this formula using the data in the table on page 1, we see that all the five Platonic solids satisfy this formula. In fact, this formula holds irrespective of whether the given polyhedron is regular.

Suppose that a Platonic solid has F faces, each of which is a regular k-gon. Suppose also that every vertex is adjacent to p faces. First of all, let us determine the number E of edges of the polyhedron. Each face of the polyhedron gives rise to k edges, but each edge is shared by two faces. Hence

$$E = \frac{\text{number of faces} \times \text{number of sides on each face}}{2} = \frac{Fk}{2}.$$

Next, let us determine the number V of vertices of the polyhedron. Each face of the polyhedron gives rise to k vertices, but each vertex is shared by p faces. Hence

$$V = \frac{\text{number of faces} \times \text{number of vertices on each face}}{p} = \frac{Fk}{p}.$$

Euler's formula then becomes

$$F + \frac{Fk}{p} - \frac{Fk}{2} = 2, \qquad \text{or} \qquad F\left(1 + \frac{k}{p} - \frac{k}{2}\right) = 2.$$

Since F is a positive quantity, it follows that

$$1 + \frac{k}{p} - \frac{k}{2} > 0, \qquad \text{or} \qquad 1 + \frac{k}{p} > \frac{k}{2}.$$

Multiplying both sides by $2/k$ now gives

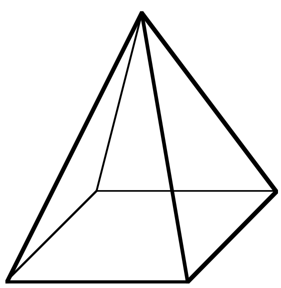
$$(*) \qquad \frac{2}{k} + \frac{2}{p} > 1.$$

But then we note that $k \geq 3$, as a polygon has at least three sides. On the other hand, each vertex must meet at least three faces, so $p \geq 3$ also. It is now not difficult to check that the only positive integers $k \geq 3$ and $p \geq 3$ that satisfy the inequality $(*)$ are given as follows:

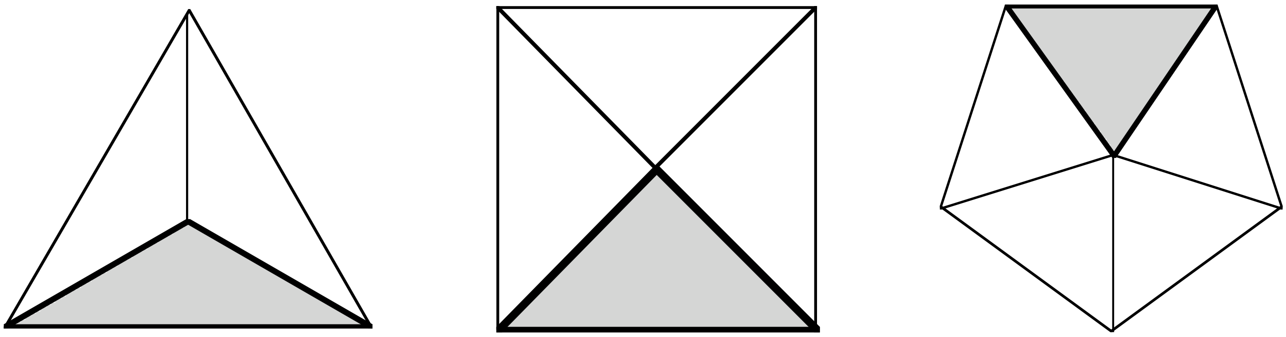
k	p	polyhedron
3	3	tetrahedron
3	4	octahedron
3	5	dodecahedron
4	3	hexahedron
5	3	icosahedron

We conclude our discussion by discussing how we may visualize four of the Platonic solids.

First of all, we are all familiar with the pyramid, a structure with a square base and triangular sloped faces. These triangles are isosceles.

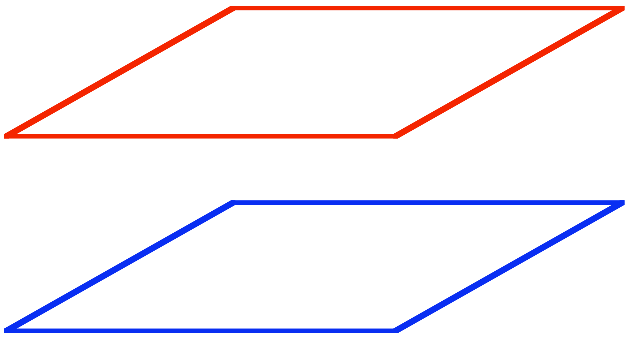


Now there is no reason why the base should be a square. We can start the base with an equilateral triangle, a square or a regular pentagon. Then the views of the triangular, square and pentagonal pyramids from above are captured in the pictures below. In each case, a sloped triangular face is highlighted. This face is an isosceles triangle.

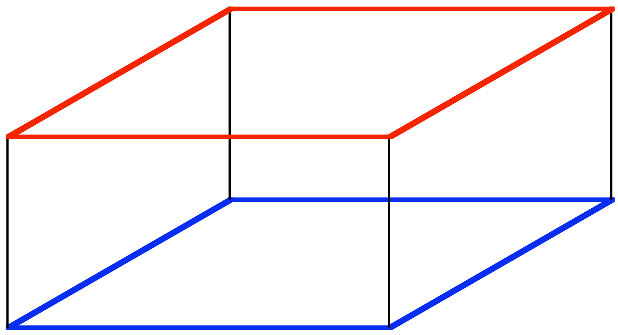


Indeed, by choosing the heights of these pyramids suitably, we can ensure that these sloped triangular faces are in fact equilateral triangles. Let us call these pyramids regular. It is then easy to see that a regular triangular pyramid is a regular tetrahedron. On the other hand, if we take two regular square pyramids, turn one of them upside down and glue the two square faces together, we obtain immediately the regular octahedron.

Next we discuss the idea of a prism. Let us start by taking two square faces of the same size, then placing them horizontally, one precisely above the other as shown.

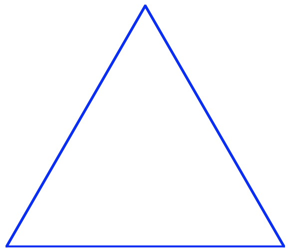


We then join corresponding corners of these squares by vertical line segments as shown.

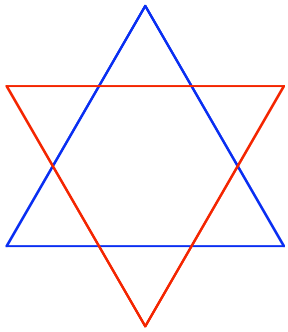


Note that we now have four vertical faces, made up of rectangles of the same size. Now there is no reason why the top and the base should be squares. We can start with two regular k-gons of the same size and place them horizontally, one precisely above the other. On joining corresponding corners of these regular k-gons by vertical line segments, we obtain k vertical faces, made up of rectangles of the same size. Indeed, by choosing the distance between the two horizontal regular k-gons suitably, we can ensure that these vertical rectangular faces are in fact squares. In this case, we call the object a regular k-gonal prism. It is then easy to see that a regular 4-gonal, or square, prism is a regular hexahedron (cube).

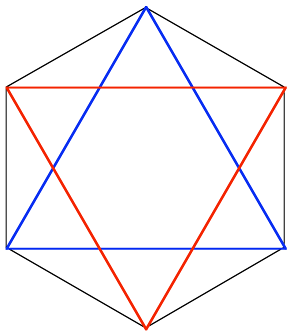
Lastly we discuss the idea of an antiprism. We begin our discussion by starting with an equilateral triangle placed horizontally, shown below in blue.



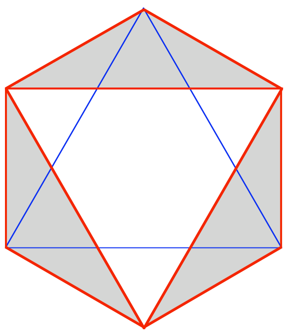
We then take another triangle of the same size and shape, rotate it suitably and then place it horizontally and some distance above the original one. This second triangle is shown below in red.

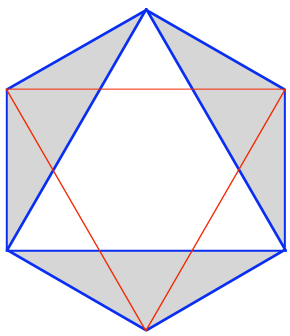


We now join each vertex of the red triangle to the two closest vertices of the blue triangle by non-horizontal line segments as shown below.

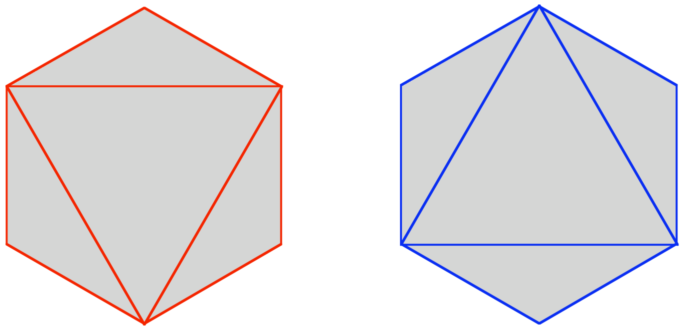


These non-horizontal line segments joining the vertices create six isosceles triangles, three pointing upwards and three pointing downwards. The three isosceles triangles pointing downwards each has a common edge with the red equilateral triangle at the top. The view from above of these three isosceles triangles pointing downwards is shown below on the left. On the other hand, the three isosceles triangles pointing upwards each has a common edge with the blue equilateral triangle at the base. The (see-through) view from above of these three isosceles triangles pointing upwards is shown below on the right.

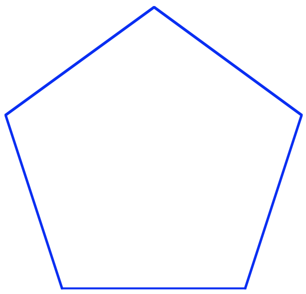




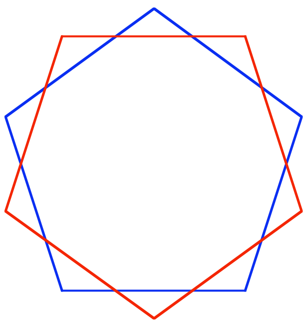
Indeed, by choosing the distance between the two horizontal equilateral triangles suitably, we can ensure that these six isosceles triangles are in fact equilateral. We then end with a polyhedron with eight equilateral triangular faces. This is the regular octahedron again. The four equilateral triangular faces visible from above are shown below on the left, while the four equilateral triangular faces visible from below are shown below on the right.



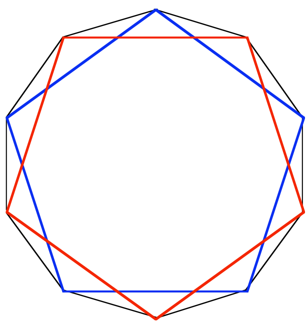
We now vary our discussion by starting with a regular pentagon placed horizontally, shown below in blue.



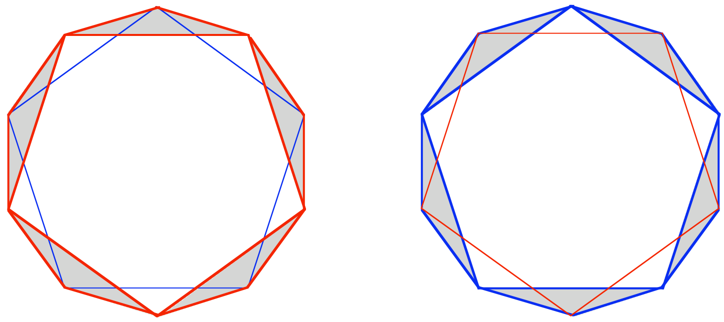
We then take another pentagon of the same size and shape, rotate it suitably and then place it horizontally and some distance above the original one. This second pentagon is shown below in red.



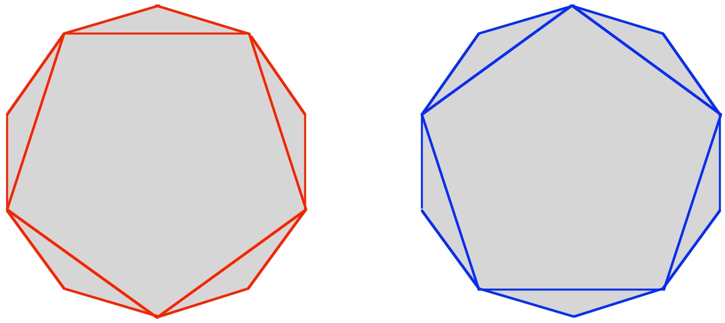
We now join each vertex of the red pentagon to the two closest vertices of the blue pentagon by non-horizontal line segments as shown below.



These non-horizontal line segments joining the vertices create ten isosceles triangles, five pointing upwards and five pointing downwards. The five isosceles triangles pointing downwards each has a common edge with the red regular pentagon at the top. The view from above of these five isosceles triangles pointing downwards is shown below on the left. On the other hand, the five isosceles triangles pointing upwards each has a common edge with the blue regular pentagon at the base. The (see-through) view from above of these five isosceles triangles pointing upwards is shown below on the right.



Indeed, by choosing the distance between the two horizontal regular pentagons suitably, we can ensure that these ten isosceles triangles are in fact equilateral. We then end with a polyhedron with two horizontal pentagonal faces and ten equilateral triangular faces. This is known as the regular pentagonal antiprism. The pentagonal face and the five equilateral triangular faces visible from above are shown below on the left, while the pentagonal face and the five equilateral triangular faces visible from below are shown below on the right.



Finally, we take two regular pentagonal pyramids with bases the same size as our two regular pentagons. We put one on top, and the other one upside down at the bottom. The two regular pentagonal faces of the regular pentagonal antiprism are now replaced by ten equilateral triangular faces, of the same size as the ten equilateral triangular faces of the regular pentagonal antiprism. Hence we obtain a polyhedron with twenty equilateral triangular faces. This is the regular isocahedron. The ten equilateral triangular faces visible from above are shown below on the left, while the ten equilateral triangular faces visible from below are shown below on the right.