Davenport's theorem in the theory of irregularities of point distribution

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ABSTRACT

We study distributions $\mathcal{D}_N$ of N points in the unit square U^2 with a minimal order of the L_2-discrepancy $\mathcal{L}_2[\mathcal{D}_N] < C(\log N)^{1/2}$, where the constant C is independent of N. We introduce an approach using Walsh functions that admits generalization to higher dimensions.

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1. Introduction

Suppose that $\mathcal{A}_N$ is a distribution of $N > 1$ points, not necessarily distinct, in the unit square $U^2 = [0,1)^2$. The L_2-discrepancy $\mathcal{L}_2[\mathcal{A}_N]$ is defined by

$$\mathcal{L}_2[\mathcal{A}_N] = \left(\int_{U^2} |\mathcal{L}[\mathcal{A}_N; Y]|^2 \, \mathrm{d}Y \right)^{1/2},$$

where for every $Y = (y_1, y_2) \in U^2$, the local discrepancy $\mathcal{L}[\mathcal{A}_N; Y]$ is given by

$$\mathcal{L}[\mathcal{A}_N; Y] = \#(\mathcal{A}_N \cap B_Y) - Ny_1y_2. \tag{1.1}$$

Here $B_Y = [0, y_1) \times [0, y_2) \subseteq U^2$ is a rectangle of area y_1y_2, while $\#(\mathcal{S})$ denotes the number of points of a set $\mathcal{S}$, counted with multiplicity.

The following results are classical.

Theorem 1. (Roth [**11**]) *There exists a positive absolute constant c such that for any distribution $\mathcal{A}_N$ of N points in the unit square U^2, we have*

$$\mathcal{L}_2[\mathcal{A}_N] > c(\log N)^{1/2}.$$

Theorem 2. (Davenport [**4**]) *There exists a positive absolute constant C such that for every natural number $N > 1$, there exist distributions $\mathcal{B}_N$ of N points in the unit square U^2 such that*

$$\mathcal{L}_2[\mathcal{B}_N] < C(\log N)^{1/2}.$$

Theorem 1 can be easily generalized to higher dimensions, with a lower bound $c_n(\log N)^{\frac{1}{2}(n-1)}$ for the n-dimensional analogue. Theorem 2 can also be extended to higher dimensions, as shown by Roth [**14**], with an upper bound of the form $C_n(\log N)^{\frac{1}{2}(n-1)}$ for the n-dimensional analogue. However, Roth's argument involves considerable extra difficulties.

On the one hand, Davenport's technique in [**4**] currently admits no generalization to higher dimensions. Indeed, a proof of the 3-dimensional analogue of Theorem 2 along the lines of Davenport's argument would require the falsity of the celebrated conjecture of Littlewood on diophantine approximation. On the other hand, while the van der Corput sequence enables one to give the best possible upper bound for the local discrepancy (1.1), it is insufficient on its own to even give a proof of Theorem 2, let alone any higher dimensional analogue. In fact, we shall show in Theorem 3 that the van der Corput sequence gives an estimate of higher order of magnitude than for the integral in Theorem 2.

In [**14**], Roth used the van der Corput sequence as generalized by Halton [**8**] and Hammersley [**9**]. He then introduced a powerful probabilistic argument to obtain higher dimensional analogues of Theorem 2, at the expense of the explicitness of the point sets $\mathcal{B}_N$ involved. Indeed, there is currently in the literature no explicit construction of point sets $\mathcal{B}_N$ which satisfy any higher dimensional analogue of Theorem 2.

The starting point of our investigation in this paper is to examine many of the ideas in the various proofs of Theorem 2. While Davenport's original proof gives an explicit point set $\mathcal{B}_N$, all subsequent proofs, by Roth [**12**,**13**,**14**], Chen [**2**], Dobrovol'skiĭ [**5**], Skriganov [**16**,**17**], and Beck and Chen [**1**], are probabilistic in nature, and do not give any explicit point sets. We shall indicate briefly how the idea of reflection introduced by Davenport in [**4**] can be combined with van der Corput point sets to give an explicit proof of Theorem 2.

We shall also discuss a new group theoretic approach to the problem, arising from the observation that certain van der Corput point sets have a very nice group structure. It is then natural to use the group characters, which turn out to be the Walsh functions. We shed new light on why the van der Corput sequence on its own is unsuitable for proving Theorem 2. We also combine this with Davenport's reflection principle to give another explicit proof of Theorem 2. While this approach does not give any new results in dimension 2, its major benefit is that it admits generalization to higher dimensions, to be discussed in a forthcoming paper [**3**].

The paper is organized as follows. In Section 2, we shall highlight the idea of reflection crucial to Davenport's proof of Theorem 2. In Section 3, we shall introduce the van der Corput point sets, state a theorem to show that they are insufficient to give a proof of Theorem 2, and indicate briefly how they can be modified to give proofs of Theorem 2. In Section 4, we introduce the Walsh functions, and combine them with the van der Corput point sets. In Section 5, we study more carefully the idea of reflection. We then combine our ideas in Section 6 to sketch a new explicit proof of Theorem 2. In view of our forthcoming paper [**3**], the discussion in Section 6 is restricted to a very brief sketch, as the details are technically complicated. Finally, we indicate the limitations of the van der Corput point sets in Section 7.

For convenience, $\mathbb{N}$ denotes the set of all positive integers, $\mathbb{N}_0$ denotes the set of all non-negative integers, $\mathbb{Q}$ denotes the set of all rational numbers, and $\mathbb{R}$ denotes the set of all real numbers. For any $\beta \in \mathbb{R}$, we let $\{\beta\}$ denote the fractional part of β, $\phi(\beta) = \{\beta\} - 1/2$ when $\beta \notin \mathbb{Z}$, $\phi(\beta) = 0$ when $\beta \in \mathbb{Z}$, and $e(\beta) = \mathrm{e}^{2\pi\mathrm{i}\beta}$. If $\mathcal{S}$ is a set, then $\chi_{\mathcal{S}}$ denotes its characteristic function. If $\mathcal{S}$ is a finite set, then $\#(\mathcal{S})$ denotes the number of elements of $\mathcal{S}$.

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2. The Ideas of Davenport

In this section, we give a brief sketch of Davenport's ideas in [**4**].

Consider a lattice Λ on the plane generated by the two vectors $(1,0)$ and $(\theta,1)$, where θ is an irrational number. Suppose that M is a positive integer. We are interested in the set $\mathcal{Q}$ which contains precisely the M points of Λ that fall into the rectangle $[0,1) \times [0,M)$. Clearly

$$\mathcal{Q} = \{(\{\theta n\}, n) : 0 \leq n \leq M-1\}.$$

Consider next a rectangle of the form

$$R(y_1, y_2) = [0, y_1) \times [0, y_2) \subseteq [0,1) \times [0, M),$$

where y_1 is arbitrary and y_2 is an integer. Then it is not difficult to show that

$$\#(\mathcal{Q} \cap R(y_1, y_2)) - y_1 y_2 = \sum_{n=0}^{y_2-1} (\phi(\theta n - y_1) - \phi(\theta n))$$

for all but a finite number of values of y_1 in the interval $[0,1)$. This has Fourier expansion

$$\sum_{m \neq 0} \left(\frac{1 - e(-m y_1)}{2\pi \mathrm{i} m} \right) \left(\sum_{n=0}^{y_2-1} e(\theta n m) \right). \tag{2.1}$$

Ideally, we would like to square the expression (2.1) and integrate with respect to y_1 over the interval $[0,1)$. Unfortunately, the term 1 in the numerator $1 - e(-my_1)$ proves to be a nuisance. In order to overcome this difficulty, consider the lattice Λ' on the plane generated by the two vectors $(1,0)$ and $(-\theta, 1)$. Then

$$\mathcal{Q}' = \{(\{-\theta n\}, n) : 0 \leq n \leq M-1\}$$

is the set which contains precisely the M points of Λ' that fall into the rectangle $[0,1)\times[0,M)$. If we now consider the $2M$ points of $\mathcal{Q}\cup\mathcal{Q}'$ that fall into this rectangle, then it is not difficult to show that

$$\#((\mathcal{Q}\cup\mathcal{Q}')\cap R(y_1,y_2))-2y_1y_2=\sum_{n=0}^{y_2-1}(\phi(\theta n-y_1)-\phi(\theta n+y_1))$$

for all but a finite number of values of y_1 in the interval $[0,1)$, and has Fourier expansion

$$\sum_{m\neq 0}\left(\frac{e(my_1)-e(-my_1)}{2\pi i m}\right)\left(\sum_{n=0}^{y_2-1}e(\theta nm)\right).$$

It follows from Parseval's theorem that

$$\int_0^1|\#((\mathcal{Q}\cup\mathcal{Q}')\cap R(y_1,y_2))-2y_1y_2|^2\,\mathrm{d}y_1\ll\sum_{m=1}^{\infty}\frac{1}{m^2}\left|\sum_{n=0}^{y_2-1}e(\theta nm)\right|^2. \tag{2.2}$$

Suppose now that the irrational number θ has a continued fraction expansion with bounded partial quotients. Then one can show that the sum on the right hand side of (2.2) is bounded above by a constant multiple of $\log(2M)$. On the other hand, relaxing the restriction that y_2 is an integer introduces an error of $O(1)$, where the implicit constant depends at most on θ. It follows trivially that

$$\int_0^M\int_0^1|\#((\mathcal{Q}\cup\mathcal{Q}')\cap R(y_1,y_2))-2y_1y_2|^2\,\mathrm{d}y_1\mathrm{d}y_2\ll M\log(2M),$$

where the implicit constant depends at most on θ. Rescaling in the y_2-direction by a factor $1/M$, we see that the set

$$\mathcal{P}=\{(\{\pm\theta n\},n/M):0\leq n\leq M-1\}$$

of $N=2M$ points in $[0,1)^2$ satisfies the requirements of Davenport's theorem.

Note that the lattices Λ and Λ' are symmetric across the vertical line $2y_1=1$.

3. Van der Corput Sets

Suppose that $s\in\mathbb{N}$. We now consider the van der Corput set of 2^s points in U^2, given by

$$\mathcal{P}(s)=\left\{\left(\sum_{i=1}^{s}a_i2^{-i},\sum_{i=1}^{s}a_{s+1-i}2^{-i}\right):a_1,a_2\ldots,a_s\in\{0,1\}\right\}.$$

The following result shows that this set is not good enough give the minimal order of the L_2-discrepancy.

Theorem 3. *For every $s\in\mathbb{N}$, we have*

$$\int_{U^2}|\#(\mathcal{P}(s)\cap B_Y)-2^sy_1y_2|^2\,\mathrm{d}Y=2^{-6}s^2+O(s).$$

We remark here that a lower bound without the specific constant 2^{-6} was given by Matoušek; see Section 2.2 of [**10**]. We therefore only give a very brief sketch of the proof in Section 9 to indicate how this constant arises. Here we shall discuss briefly how this set can be modified to give a proof of Theorem 2.

Definition. Suppose that $s\in\mathbb{N}_0$. By an s-box, we mean a rectangle of the type

$$[m_12^{-i_1},(m_1+1)2^{-i_1})\times[m_22^{-i_2},(m_2+1)2^{-i_2})\subseteq U^2,$$

where $m_1, m_2, i_1, i_2 \in \mathbb{N}_0$ satisfy the condition $i_1 + i_2 = s$, so that the rectangle has area 2^{-s}.

The following simple observation follows immediately from the definition of van der Corput sets and the definition of s-boxes.

Lemma 3A. *Suppose that $s \in \mathbb{N}_0$. Then every s-box contains precisely one point of the van der Corput set $\mathcal{P}(s)$.*

It is convenient to rescale in the y_2-direction. Accordingly, we consider the set

$$\mathcal{Q}(s) = \left\{ \left(\sum_{i=1}^{s} a_i 2^{-i}, \sum_{i=1}^{s} a_{s+1-i} 2^{s-i} \right) : a_1, a_2 \ldots, a_s \in \{0,1\} \right\} \subset [0,1) \times [0, 2^s).$$

Suppose that $y_1 \in [0,1)$ is a fixed integer multiple of 2^{-s}. Then it can be shown that for almost all real numbers $y_2 \in [0, 2^s)$, we have

$$\#(\mathcal{Q}(s) \cap ([0, y_1) \times [0, y_2))) - y_1 y_2 = \sum_{i \in \mathcal{I}} \left(c_i + \phi\left(\frac{z_i - y_2}{L_i} \right) \right),$$

where $\mathcal{I}$ is a finite set of indices depending only on y_1, and where the integer L_i is a divisor of 2^s for every $i \in \mathcal{I}$. To study L_2-discrepancy, we therefore need to consider sums of the form

$$\sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} \left(c_i + \phi\left(\frac{z_i - y_2}{L_i} \right) \right) \left(c_j + \phi\left(\frac{z_j - y_2}{L_j} \right) \right). \tag{3.1}$$

Integrating y_2 over the interval $[0, 2^s)$, each summand in (3.1) gives rise to an integral of the type

$$\int_0^{2^s} \left(c_i + \phi\left(\frac{z_i - y_2}{L_i} \right) \right) \left(c_j + \phi\left(\frac{z_j - y_2}{L_j} \right) \right) \mathrm{d}y_2 = 2^s \left(c_i c_j + O\left(\frac{(L_i, L_j)^2}{L_i L_j} \right) \right),$$

where, for every $i, j \in \mathcal{I}$, the term (L_i, L_j) denotes the greatest common divisor of L_i and L_j. Unfortunately, the term $c_i c_j$ prevents any further progress.

Remark. The constants c_i can be calculated explicitly. A careful analysis using the above will lead to a proof of Theorem 3.

The idea of Roth in [**14**] is to introduce a translation variable t in the y_2-direction, and consider instead point sets of the type

$$\mathcal{Q}(s; t) = \{(x_1, x_2 + t) : (x_1, x_2) \in \mathcal{Q}(s)\},$$

where the addition $x_2 + t$ is modulo 2^s. Then

$$\#(\mathcal{Q}(s; t) \cap ([0, y_1) \times [0, y_2)) - y_1 y_2 = \sum_{i \in \mathcal{I}} \left(\phi\left(\frac{z_i - y_2 + t}{L_i} \right) - \phi\left(\frac{z_i' - y_2 + t}{L_i} \right) \right).$$

Squaring the sum on the right hand side and integrating over t, we see that every summand now gives rise to four integrals of the type

$$\int_0^{2^s} \phi\left(\frac{z_i - y_2 + t}{L_i} \right) \phi\left(\frac{z_j - y_2 + t}{L_j} \right) \mathrm{d}t = O\left(2^s \frac{(L_i, L_j)^2}{L_i L_j} \right).$$

Alternatively, we consider the reflected set

$$\mathcal{Q}'(s) = \{(x_1, 2^s - x_2) : (x_1, x_2) \in \mathcal{Q}(s)\},$$

where the subtraction $2^s - x_2$ is modulo 2^s. Then it can be shown that for almost all real numbers $y_2 \in [0, 2^s)$, we have

$$\#(\mathcal{Q}'(s) \cap ([0, y_1) \times [0, y_2)) - y_1 y_2 = \sum_{i \in \mathcal{I}} \left(-c_i + \phi\left(\frac{z_i' - y_2}{L_i}\right)\right),$$

so that the combined local discrepancy is given by

$$\#((\mathcal{Q}(s) \cup \mathcal{Q}'(s)) \cap ([0, y_1) \times [0, y_2)) - 2y_1 y_2 = \sum_{i \in \mathcal{I}} \left(\phi\left(\frac{z_i - y_2}{L_i}\right) + \phi\left(\frac{z_i' - y_2}{L_i}\right)\right).$$

Squaring the sum on the right hand side and integrating over y_2, we see that every summand now gives rise to four integrals of the type

$$\int_0^{2^s} \phi\left(\frac{z_i - y_2}{L_i}\right) \phi\left(\frac{z_j - y_2}{L_j}\right) \mathrm{d}t = O\left(2^s \frac{(L_i, L_j)^2}{L_i L_j}\right).$$

We can therefore conclude that the set $\mathcal{Q}(s) \cup \mathcal{Q}'(s)$ of 2^{s+1} points satisfies the inequality

$$\int_0^{2^s} \int_0^1 |\#((\mathcal{Q}(s) \cup \mathcal{Q}'(s)) \cap ([0, y_1) \times [0, y_2))) - 2y_1 y_2|^2 \, \mathrm{d}y_1 \mathrm{d}y_2 = O(2^s s).$$

This leads to explicit constructions satisfying the conclusion of Theorem 2. We omit the details here.

4. The Unit Interval and Walsh Functions

For any $s \in \mathbb{N}_0$, let $\mathbb{Q}(2^s) = \{m2^{-s} : m = 0, 1, \ldots, 2^s - 1\}$, and let

$$\mathbb{Q}(2^\infty) = \bigcup_{s=0}^{\infty} \mathbb{Q}(2^s)$$

denote the binary rational numbers.

Observe that any $x \in [0, 1)$ can be represented in the form

$$x = \sum_{i=1}^{\infty} \eta_i(x) 2^{-i}, \tag{4.1}$$

where the coefficients $\eta_i(x) \in \{0, 1\}$ for every $i \in \mathbb{N}$. This representation is unique if we agree that the series in (4.1) is finite for every $x \in \mathbb{Q}(2^\infty)$. For any two elements $x, y \in \mathbb{Q}(2^\infty)$, we can write

$$x \oplus y \in \mathbb{Q}(2^\infty) \tag{4.2}$$

by setting

$$\eta_i(x \oplus y) = \eta_i(x) + \eta_i(y) \pmod 2$$

for every $i \in \mathbb{N}$. It is easy to see that with respect to the operation (4.2), each set $\mathbb{Q}(2^s)$ forms a finite abelian group, while the set $\mathbb{Q}(2^\infty)$ forms an infinite abelian group. It is well known that the characters of these groups are the Walsh functions.

Any $\ell \in \mathbb{N}_0$ can be written uniquely in the form

$$\ell = \sum_{i=1}^{\infty} \lambda_i(\ell) 2^{i-1}, \tag{4.3}$$

where the coefficients $\lambda_i(\ell) \in \{0, 1\}$ for every $i \in \mathbb{N}$. For every real number $x \in [0, 1)$ of the form (4.1), we consider the Walsh function

$$w_\ell(x) = (-1)^{\sum_{i=1}^{\infty} \lambda_i(\ell) \eta_i(x)}. \tag{4.4}$$

A detailed study of these functions can be found in [**7**] and [**15**].

Since (4.3) is essentially a finite sum, the function $w_\ell(x)$ is well defined, and takes the values ± 1. We have $w_0(x) = 1$ for every $x \in [0,1)$, and

$$\int_0^1 w_\ell(y)\,\mathrm{d}y = 0$$

for every $\ell \in \mathbb{N}$. In other words, apart from the one exception $w_0(x)$, all Walsh functions have zero mean over the unit interval. Of particular importance is the fact that the Walsh functions form an orthonormal basis of $L_2([0,1))$.

It is easy to see that for every $s \in \mathbb{N}$, the van der Corput set $\mathcal{P}(s)$ of 2^s points in U^2 forms a group under the operation (4.2) for each coordinate. On the other hand, we can combine Lemma 3A with properties of the Walsh functions to establish a number of results. In the following two lemmas, we use the notation $X = (x_1, x_2)$.

Lemma 4A. *Suppose that $\mathcal{P}(s)$ is the van der Corput set of 2^s points in U^2. Then for any $i, q \in \mathbb{N}_0$ satisfying $i < s$ and $0 \le q < 2^i$, we have*

$$\sum_{X \in \mathcal{P}(s)} w_{2^i \oplus q}(x_1) = 0.$$

Lemma 4B. *Suppose that $\mathcal{P}(s)$ is the van der Corput set of 2^s points in U^2. Then for any $i_1, i_2 \in \mathbb{N}_0$, the following assertions hold:*

(i) Suppose that $i_1 + i_2 < s - 1$. Then for any $q_1, q_2 \in \mathbb{N}_0$ satisfying $0 \le q_1 < 2^{i_1}$ and $0 \le q_2 < 2^{i_2}$, we have

$$\sum_{X \in \mathcal{P}(s)} w_{2^{i_1} \oplus q_1}(x_1) w_{2^{i_2} \oplus q_2}(x_2) = 0.$$

(ii) We have

$$\sum_{X \in \mathcal{P}(s)} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) \chi_{[0,2^{-i_1})}(x_1) \chi_{[0,2^{-i_2})}(x_2) = \begin{cases} 0 & \text{if } i_1 + i_2 \le s-2, \\ 1 & \text{if } i_1 + i_2 \ge s, \end{cases}$$

and

$$\left| \sum_{X \in \mathcal{P}(s)} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) \chi_{[0,2^{-i_1})}(x_1) \chi_{[0,2^{-i_2})}(x_2) \right| \le 2 \quad \text{if } i_1 + i_2 = s-1.$$

(iii) We have

$$\left| \sum_{X \in \mathcal{P}(s)} w_{2^{i_1}}(x_1) \chi_{[0,2^{-i_1})}(x_1) w_{2^{i_2}}(x_2) \right| \le \max\{1, 2^{s-i_1}\}.$$

5. Davenport Reflection

Suppose that $\mathcal{P}(s)$ is the van der Corput set of 2^s points in U^2. Consider the mapping

$$\Theta : U^2 \to [0,1]^2 : (x_1, x_2) \mapsto (|2x_1 - 1|, |2x_2 - 1|). \tag{5.1}$$

We shall study the point set $\mathcal{P}^*(s) = \Theta(\mathcal{P}(s))$, where the points are counted with multiplicity, so that $\mathcal{P}^*(s)$ is a distribution of 2^s points in $[0,1]^2$. Note that apart from the point $\Theta(0,0) = (1,1)$, all other points of $\mathcal{P}^*(s)$ are in $[0,1)^2$.

Suppose that $Y = (y_1, y_2) \in [0, 1/2)^2$. Then the two rectangles

$$B_Y^{\mathrm{mod}} = (y_1, 1-y_1) \times (y_2, 1-y_2)$$

and

$$B_{\Theta(Y)} = [0, 1-2y_1) \times [0, 1-2y_2)$$

have the same area $(1-2y_1)(1-2y_2)$. Furthermore, for every point $X \in \mathcal{P}(s)$, we have

$$X \in B_Y^{\mathrm{mod}} \qquad \text{if and only if} \qquad \Theta(X) \in B_{\Theta(Y)}.$$

Hence the two quantities

$$\mathcal{L}^{\mathrm{mod}}[\mathcal{P}(s); Y] = \#(\mathcal{P}(s) \cap B_Y^{\mathrm{mod}}) - 2^s(1-2y_1)(1-2y_2)$$

and

$$\mathcal{L}[\mathcal{P}^*(s); \Theta(Y)] = \#(\mathcal{P}^*(s) \cap B_{\Theta(Y)}) - 2^s(1-2y_1)(1-2y_2)$$

are equal, and it follows via the substitution $V = \Theta(Y)$ that

$$\int_{U^2} |\mathcal{L}[\mathcal{P}^*(s); V]|^2 \,\mathrm{d}V = 4 \int_{[0,1/2)^2} |\mathcal{L}^{\mathrm{mod}}[\mathcal{P}(s); Y]|^2 \,\mathrm{d}Y. \tag{5.2}$$

Theorem 2 can easily be deduced from the following result.

Lemma 5A. *Suppose that $\mathcal{P}(s)$ is the van der Corput set of 2^s points in U^2. Then*

$$\int_{[0,1/2)^2} |\mathcal{L}^{\mathrm{mod}}[\mathcal{P}(s); Y]|^2 \,\mathrm{d}Y \le 300s.$$

6. Sketch of the Proof of Lemma 5A

For the remainder of this paper, $\mathcal{P} = \mathcal{P}(s)$ denotes the van der Corput set of 2^s points in U^2. Since

$$\#(\mathcal{P} \cap B_Y^{\mathrm{mod}}) = \sum_{X \in \mathcal{P}} \chi_{B_Y^{\mathrm{mod}}}(X) = \sum_{X \in \mathcal{P}} \chi_{(y_1, 1-y_1)}(x_1)\chi_{(y_2,1-y_2)}(x_2),$$

our first step is naturally to describe the characteristic function $\chi_{(y,1-y)}(x)$ in terms of Walsh functions.

Suppose that $y \in [0,1)$ is fixed. Since the Walsh functions form an orthonormal basis of $L_2([0,1))$, it follows that for every $x \in [0,1)$, we can write

$$\chi_{[0,y)}(x) \simeq \sum_{\ell=0}^{\infty} \tilde{\chi}_\ell(y) w_\ell(x), \tag{6.1}$$

where the symbol $\simeq$ denotes that the series (6.1) converges in the L_2-norm and where, for every $\ell \in \mathbb{N}_0$, we have

$$\tilde{\chi}_\ell(y) = \int_0^1 \chi_{[0,y)}(x) w_\ell(x) \,\mathrm{d}x = \int_0^y w_\ell(x) \,\mathrm{d}x. \tag{6.2}$$

In particular, we have

$$\tilde{\chi}_0(y) = \int_0^y w_0(x) \,\mathrm{d}x = y. \tag{6.3}$$

On the other hand, there exists a unique $m \in \mathbb{N}_0$ such that $y \in [m2^{-s}, (m+1)2^{-s})$. Consider the function

$$\chi_{[0,y)}^{(s)}(x) = \begin{cases} 1 & \text{if } 0 \le x < m2^{-s}, \\ 2^s y - m & \text{if } m2^{-s} \le x < (m+1)2^{-s}, \\ 0 & \text{if } (m+1)2^{-s} \le x < 1, \end{cases}$$

where

$$2^s y - m = 2^s \int_{m2^{-s}}^{(m+1)2^{-s}} \chi_{[0,y)}(x) \,\mathrm{d}x$$

represents the average value of $\chi_{[0,y)}(x)$ in the interval $[m2^{-s},(m+1)2^{-s})$. It is well known that

$$\chi^{(s)}_{[0,y)}(x)=\sum_{\ell=0}^{2^s-1}\tilde{\chi}_\ell(y)w_\ell(x). \tag{6.4}$$

For details, see Section 2.8 of [**7**]. It is easy to see that $|\chi_{[0,y)}(x)-\chi^{(s)}_{[0,y)}(x)|$ is equal to 0 whenever $x\notin[m2^{-s},(m+1)2^{-s})$, and is at most 1 always.

The evaluation of the integral (6.2) was studied by Fine [**6**]. For every $\ell\in\mathbb{N}$, we have the identity

$$\tilde{\chi}_\ell(y)=\frac{1}{4}2^{-\nu(\ell)}\left(w_{\ell\oplus2^{\nu(\ell)}}(y)-\sum_{j=1}^{\infty}2^{-j}w_{\ell\oplus2^{\nu(\ell)+j}}(y)\right), \tag{6.5}$$

where $\nu(\ell)\in\mathbb{N}_0$ denotes the unique integer satisfying $2^{\nu(\ell)}\le\ell<2^{\nu(\ell)+1}$. For the special case when $\ell=2^i$, where $i\in\mathbb{N}_0$, we have $\ell\oplus2^{\nu(\ell)}=0$, and so the function $\tilde{\chi}_\ell(y)$ has non-zero mean value over the interval $[0,1)$. We therefore introduce the quantity

$$\delta_\ell=\begin{cases}1 & \text{if } \ell=2^i \text{ for some } i\in\mathbb{N}_0,\\ 0 & \text{if } \ell\ne2^i \text{ for any } i\in\mathbb{N}_0,\end{cases}$$

and study instead the function

$$\Omega_\ell(y)=w_{\ell\oplus2^{\nu(\ell)}}(y)-\delta_\ell-\sum_{j=1}^{\infty}2^{-j}w_{\ell\oplus2^{\nu(\ell)+j}}(y). \tag{6.6}$$

The need to handle the extra term δ_ℓ is the reason for introducing the reflection mapping (5.1). Suppose instead that $y\in[0,1/2)$. Then for almost all $x\in[0,1)$, we have

$$\chi_{(y,1-y)}(x)=\chi_{[0,1-y)}(x)-\chi_{[0,y)}(x).$$

Now let

$$\chi^{(s)}_{(y,1-y)}(x)=\chi^{(s)}_{[0,1-y)}(x)-\chi^{(s)}_{[0,y)}(x).$$

Then almost always, we have

$$\begin{aligned}\chi^{(s)}_{(y,1-y)}(x)&=1-2y+\sum_{\ell=1}^{2^s-1}(\tilde{\chi}_\ell(1-y)-\tilde{\chi}_\ell(y))w_\ell(x)\\&=1-2y+\frac{1}{4}\sum_{\ell=1}^{2^s-1}2^{-\nu(\ell)}(\Omega_\ell(1-y)-\Omega_\ell(y))w_\ell(x).\end{aligned}$$

On the other hand, it is easy to see that if we write

$$\chi^{(s)}_{B_Y^{\mathrm{mod}}}(X)=\chi^{(s)}_{(y_1,1-y_1)}(x_1)\chi^{(s)}_{(y_2,1-y_2)}(x_2),$$

then $|\chi_{B_Y^{\mathrm{mod}}}(X)-\chi^{(s)}_{B_Y^{\mathrm{mod}}}(X)|$ is equal to 0 if X does not belong to four s-boxes, and is at most 2 always. It follows that if we write

$$\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y]=\sum_{X\in\mathcal{P}}\chi^{(s)}_{B_Y^{\mathrm{mod}}}(X)-2^s(1-2y_1)(1-2y_2), \tag{6.7}$$

then the inequality $|\mathcal{L}^{\mathrm{mod}}[\mathcal{P};Y]|\le|\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y]|+8$ holds for almost all $Y\in[0,1/2)^2$, so that

$$\int_{[0,1/2)^2}|\mathcal{L}^{\mathrm{mod}}[\mathcal{P};Y]|^2\,\mathrm{d}Y\le2\int_{[0,1/2)^2}|\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y]|^2\,\mathrm{d}Y+128.$$

The proof of Lemma 5A is now reduced to finding a suitable upper bound for the integral

$$\int_{[0,1/2)^2} |\mathcal{M}^{\mathrm{mod}}[\mathcal{P};Y]|^2 \,\mathrm{d}Y.$$

We omit the long and technical calculations here.

7. Sketch of the Proof of Theorem 3

Note that it follows from (6.3)–(6.6) that for every $y \in [0,1)$, we have

$$\chi^{(s)}_{[0,y)}(x) = y + \frac{1}{4}\sum_{i=0}^{s-1} 2^{-i} w_{2^i}(x) + \frac{1}{4}\sum_{\ell=1}^{2^s-1} 2^{-\nu(\ell)}\Omega_\ell(y) w_\ell(x).$$

Using this, we can show, analogous to (6.7), that the local discrepancy $\mathcal{L}[\mathcal{P};Y]$ can be approximated by a function of the type

$$\begin{aligned}\mathcal{M}[\mathcal{P};Y] = \frac{1}{4}y_2 \sum_{i_1=0}^{s-1} 2^{-i_1} \sum_{X\in\mathcal{P}} w_{2^{i_1}}(x_1) + \frac{1}{4}y_1 \sum_{i_2=0}^{s-1} 2^{-i_2} \sum_{X\in\mathcal{P}} w_{2^{i_2}}(x_2) \\ + \frac{1}{16}\sum_{i_1=0}^{s-1}\sum_{i_2=0}^{s-1} 2^{-i_1-i_2} \sum_{X\in\mathcal{P}} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) + \mathcal{Z}[\mathcal{P};Y],\end{aligned}$$

where the function $\mathcal{Z}[\mathcal{P};Y]$ has zero mean over $Y \in U^2$ and satisfies

$$\int_{U^2} |\mathcal{Z}[\mathcal{P};Y]|^2 \,\mathrm{d}Y = O(s).$$

It can be shown that if $i_1 \le s-1$ and $i_2 \le s-1$, then

$$\sum_{X\in\mathcal{P}} w_{2^{i_1}}(x_1) w_{2^{i_2}}(x_2) = \begin{cases} 2^s & \text{if } i_1+i_2 = s-1, \\ 0 & \text{otherwise.}\end{cases}$$

Using this and Lemma 4A, we see that

$$\mathcal{M}[\mathcal{P};Y] = \frac{1}{16}\sum_{\substack{i_1=0 \\ i_1+i_2=s-1}}^{s-1}\sum_{i_2=0}^{s-1} 2^{s-i_1-i_2} + \mathcal{Z}[\mathcal{P};Y] = 2^{-3}s + \mathcal{Z}[\mathcal{P};Y].$$

This gives

$$\int_{U^2} |\mathcal{M}[\mathcal{P};Y]|^2 \,\mathrm{d}Y = 2^{-6}s^2 + O(s).$$

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