Irregularities of point distribution relative to half–planes I

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1. Introduction

Suppose that $\mathcal{P}$ is a distribution of N points in U_0, the closed disc of unit area and centred at the origin $\mathbf{0}$. For every measurable set B in $\mathbb{R}^2$, let $Z[\mathcal{P};B]$ denote the number of points of $\mathcal{P}$ in B, and write

$$D[\mathcal{P};B]=Z[\mathcal{P};B]-N\mu(B\cap U_0),$$

where μ denotes the usual measure in $\mathbb{R}^2$.

For every real number $r\in\mathbb{R}$ and every angle θ satisfying $0\le\theta\le 2\pi$, let $S(r,\theta)$ denote the closed half–plane

$$S(r,\theta)=\{\mathbf{x}\in\mathbb{R}^2:\mathbf{x}\cdot\mathbf{e}(\theta)\ge r\}.$$

Here $\mathbf{e}(\theta)=(\cos\theta,\sin\theta)$ and $\mathbf{x}\cdot\mathbf{y}$ denotes the scalar product of $\mathbf{x}$ and $\mathbf{y}$.

Roth asked the question (see Schmidt [**7**], pages 124–125) of whether

$$\inf_{|\mathcal{P}|=N}\sup_{\substack{0\le r\le\pi^{-1/2}\\0\le\theta\le 2\pi}}|D[\mathcal{P};S(r,\theta)]|\to+\infty$$

as $N\to\infty$. Here the supremum is taken over all disc–segments in U_0, and the infimum is taken over all distributions $\mathcal{P}$ of N points in U_0.

This question was answered in the affirmative by Beck [**2**], who proved in 1983 that

$$\inf_{|\mathcal{P}|=N}\sup_{\substack{0\le r\le\pi^{-1/2}\\0\le\theta\le 2\pi}}|D[\mathcal{P};S(r,\theta)]|\gg N^{1/4}(\log N)^{-7/2}.$$

Recently, Alexander [**1**] improved this to

$$\inf_{|\mathcal{P}|=N}\sup_{\substack{0\le r\le\pi^{-1/2}\\0\le\theta\le 2\pi}}|D[\mathcal{P};S(r,\theta)]|\gg N^{1/4}.$$

Beck and Alexander basically studied the L^2–norm of the discrepancy function $D[\mathcal{P};S(r,\theta)]$. The following result can be proved.

Theorem A. *For every distribution $\mathcal{P}$ of N points in U_0, we have*

$$\int_0^{2\pi}\int_0^{\pi^{-1/2}} |D[\mathcal{P};S(r,\theta)]|^2 \mathrm{d}r\mathrm{d}\theta \gg N^{1/2}.$$

This is complemented by the result below, which can be proved using probabilistic methods.

Theorem B. *For every natural number N, there exists a distribution $\mathcal{P}$ of N points in U_0 such that*

$$\int_0^{2\pi}\int_0^{\pi^{-1/2}} |D[\mathcal{P};S(r,\theta)]|^2 \mathrm{d}r\mathrm{d}\theta \ll N^{1/2}.$$

The purpose of this paper is to study the L^1–norm of the discrepancy function $D[\mathcal{P};S(r,\theta)]$. We shall prove, in particular, the following rather surprising result.

Theorem. *For every natural number $N \geq 2$, there exists a distribution $\mathcal{P}$ of N points in U_0 such that*

$$\int_0^{2\pi}\int_0^{\pi^{-1/2}} |D[\mathcal{P};S(r,\theta)]| \mathrm{d}r\mathrm{d}\theta \ll (\log N)^2.$$

Our work in this paper is in fact motivated by the case when U_0 is a square and not a disc, and only for very special values of N. In developing the method to prove the theorem above, we realized that it is possible to study the problem in far greater generality.

Let U be a convex set in $\mathbb{R}^2$ of unit area, and with centre of gravity at the origin $\mathbf{0}$. Suppose that $\mathcal{P}$ is a distribution of N points in U. For every measurable set B in $\mathbb{R}^2$, let $Z[\mathcal{P};B]$ denote the number of points of $\mathcal{P}$ in B, and write

$$D[\mathcal{P};B] = Z[\mathcal{P};B] - N\mu(B\cap U).$$

For any θ satisfying $0 \leq \theta \leq 2\pi$, let

$$R(\theta) = \sup\{r \geq 0 : S(r,\theta)\cap U \neq \emptyset\}.$$

We shall in fact prove

Main Theorem. *For every natural number $N \geq 2$, there exists a distribution $\mathcal{P}$ of N points in U such that*

$$\int_0^{2\pi}\int_0^{R(\theta)} |D[\mathcal{P};S(r,\theta)]| \mathrm{d}r\mathrm{d}\theta \ll_U (\log N)^2.$$

This work was carried out in part while the first author was Visiting Fellow at Imperial College. We thank the Science and Engineering Research Council for the financial support.

We also thank Bob Vaughan for pointing out the simple proof of Lemma 1, and the referee for his valuable comments.

2. A special case: U is a square

We first of all consider the case when U is the square $[-1/2, 1/2]^2$, and show that for every natural number N, there exists a set $\mathcal{P}$ of $4N^2 + 4N + 1$ points in U such that

$$\int_0^{2\pi} \int_0^{R(\theta)} |D[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll (\log N)^2.$$

For ease of notation, we consider the following renormalized version of the problem. Let V be the square $[-N-1/2, N+1/2]^2$. For every finite distribution $\mathcal{P}$ of points in V and every measurable set B in $\mathbb{R}^2$, let $Z[\mathcal{P}; B]$ denote the number of points of $\mathcal{P}$ in B, and write

$$E[\mathcal{P}; B] = Z[\mathcal{P}; B] - \mu(B \cap V).$$

We shall show that the set

$$\mathcal{P} = \{-N, -N+1, \ldots, -1, 0, 1, \ldots, N-1, N\}^2$$

of $4N^2 + 4N + 1$ integer lattice points in V satisfies

$$\int_0^{2\pi} \int_0^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll N(\log N)^2, \tag{1}$$

where, for every $\theta \in [0, 2\pi]$, we have $M(\theta) = (2N+1)R(\theta)$.

The line

$$T(r,\theta) = \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x} \cdot \mathbf{e}(\theta) = r\}$$

is the boundary of the half–plane $S(r,\theta)$, and can be rewritten in the form

$$x_1 \cos\theta + x_2 \sin\theta = r,$$

where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$.

Suppose that $0 \le \theta \le \pi/4$. Clearly $M(\theta) = (N+1/2)(\cos\theta + \sin\theta)$. We distinguish two cases.

Case 1: $0 \le r \le (N+1/2)(\cos\theta - \sin\theta)$. It is not difficult to see that $T(r,\theta)$ intersects the edges $\{(x_1, N+1/2) : |x_1| \le N+1/2\}$ and $\{(x_1, -N-1/2) : |x_1| \le N+1/2\}$ of V, *i.e.*, the "top" and "bottom" edges of V. Then

$$S(r,\theta) \cap V = \bigcup_{n=-N}^{N} S(n, V, r, \theta),$$

where, for every $n = -N, \ldots, 0, \ldots, N$,

$$S(n, V, r, \theta) = S(r, \theta) \cap V \cap (\mathbb{R} \times [n - 1/2, n + 1/2]).$$

Clearly

$$E[\mathcal{P}; S(r, \theta)] = \sum_{n=-N}^{N} E[\mathcal{P}; S(n, V, r, \theta)].$$

Now, for every $n = -N, \ldots, 0, \ldots, N$, we have

$$Z[\mathcal{P}; S(n, V, r, \theta)] = [N + n \tan\theta - r \sec\theta + 1]$$

and

$$\mu(S(n, V, r, \theta)) = N + n \tan\theta - r \sec\theta + 1/2,$$

so that

$$E[\mathcal{P}; S(n, V, r, \theta)] = -\psi(n \tan\theta - r \sec\theta),$$

where $\psi(z) = z - [z] - 1/2$ for every $z \in \mathbb{R}$. Hence

$$E[\mathcal{P}; S(r, \theta)] = -\sum_{n=-N}^{N} \psi(n \tan\theta - r \sec\theta).$$

Case 2: $(N+1/2)(\cos\theta - \sin\theta) \leq r \leq (N+1/2)(\cos\theta + \sin\theta)$. It is not difficult to see that $T(r, \theta)$ intersects the edges $\{(x_1, N+1/2) : |x_1| \leq N+1/2\}$ and $\{(N+1/2, x_2) : |x_2| \leq N + 1/2\}$ of V, *i.e.*, the "top" and "right" edges of V. Furthermore,

$$\begin{aligned} &T(r, \theta) \cap \{(N + 1/2, x_2) : |x_2| \leq N + 1/2\} \\ &\quad = \{(N + 1/2, -(N + 1/2) \cot\theta + r \operatorname{cosec}\theta)\}. \end{aligned}$$

Then $S(n, V, r, \theta) = \emptyset$ if $n < -(N + 1/2) \cot\theta + r \operatorname{cosec}\theta - 1/2$. On the other hand, it is trivial that $E[\mathcal{P}; S(n, V, r, \theta)] = O(1)$ always. It follows that

$$E[\mathcal{P}; S(r, \theta)] = -\sum_{\substack{n=-N \\ (*)}}^{N} \psi(n \tan\theta - r \sec\theta) + O(1),$$

where the summation is under the further restriction

$$n \geq -(N + 1/2) \cot\theta + r \operatorname{cosec}\theta. \tag{$*$}$$

Note that in Case 1, the restriction $(*)$ would become superfluous since it is weaker than the requirement $n \geq -N$. It follows that for all $r \geq 0$, we have

$$E[\mathcal{P}; S(r, \theta)] - G[\mathcal{P}; r, \theta] \ll 1,$$

where

$$G[\mathcal{P}; r, \theta] = -\sum_{\substack{n=-N \\ (*)}}^{N} \psi(n \tan\theta - r \sec\theta).$$

The function $\psi(z) = z - [z] - 1/2$ has the Fourier expansion

$$-\sum_{\nu \neq 0} \frac{e(z\nu)}{2\pi \mathrm{i} \nu},$$

so that $-\psi(n \tan\theta - r \sec\theta)$ has the Fourier expansion

$$\sum_{\nu \neq 0} \frac{e(-r\nu \sec\theta)}{2\pi \mathrm{i} \nu} e(n\nu \tan\theta).$$

It follows that the Fourier expansion of $G[\mathcal{P}; r, \theta]$ is given by

$$\sum_{\nu \neq 0} \frac{e(-r\nu \sec\theta)}{2\pi \mathrm{i} \nu} \sum_{\substack{n=-N \\ (*)}}^{N} e(n\nu \tan\theta).$$

However, the restriction $(*)$ prevents us from applying Parseval's theorem.

To overcome this difficulty, we introduce the following idea which is motivated by Roth's variation of Davenport's method (see Roth [**6**] and §3.1 of Beck and Chen [**3**]).

Let $\mathbf{y} = (y_1, y_2) \in [-1/2, 1/2]^2$. For every $\theta \in [0, \pi/4]$ and every $r \geq 1$, let

$$T(\mathbf{y}; r, \theta) = T(r + y_1 \cos\theta + y_2 \sin\theta, \theta) \tag{2}$$

and

$$S(\mathbf{y}; r, \theta) = S(r + y_1 \cos\theta + y_2 \sin\theta, \theta) \tag{3}$$

(note here that $r + y_1 \cos\theta + y_2 \sin\theta \geq 0$ always). Then

$$E[\mathcal{P}; S(\mathbf{y}; r, \theta)] = E[\mathcal{P}; S(r + y_1 \cos\theta + y_2 \sin\theta, \theta)].$$

It is not difficult to see that if we write

$$G[\mathcal{P}; \mathbf{y}; r, \theta] = -\sum_{\substack{n=-N \\ (*)}}^{N} \psi(n \tan\theta - (r + y_1 \cos\theta + y_2 \sin\theta) \sec\theta),$$

then

$$E[\mathcal{P}; S(\mathbf{y}; r, \theta)] - G[\mathcal{P}; \mathbf{y}; r, \theta] \ll \begin{cases} \cot\theta & (M(\theta) - (2N+1)\sin\theta - 1 \leq r \leq M(\theta)), \\ 1 & (\text{otherwise}), \\ N & (\text{trivially}), \end{cases}$$

so that
$$\int_0^{\pi/4}\int_1^{M(\theta)} |E[\mathcal{P};S(\mathbf{y};r,\theta)] - G[\mathcal{P};\mathbf{y};r,\theta]|\mathrm{d}r\mathrm{d}\theta \ll N \tag{4}$$
(note that $|y_1\cos\theta+y_2\sin\theta|\le 1$, so that if $r\le M(\theta)-(2N+1)\sin\theta-1$, then $T(\mathbf{y};r,\theta)$ intersects the top and bottom edges of V).

Now $G[\mathcal{P};\mathbf{y};r,\theta]$ has the Fourier expansion

$$\begin{aligned}
&\sum_{\nu\neq 0}\frac{e(-(r+y_1\cos\theta+y_2\sin\theta)\nu\sec\theta)}{2\pi\mathrm{i}\nu}\sum_{\substack{n=-N\\(*)}}^{N} e(n\nu\tan\theta)\\
&\quad=\sum_{\nu\neq 0}\frac{e(-r\nu\sec\theta)}{2\pi\mathrm{i}\nu}\sum_{\substack{n=-N\\(*)}}^{N} e((n-y_2)\nu\tan\theta)e(-y_1\nu).
\end{aligned}$$

It follows that for every $y_2\in[-1/2,1/2]$, we have, by Parseval's theorem, that

$$\begin{aligned}
\int_{-1/2}^{1/2}|G[\mathcal{P};\mathbf{y};r,\theta]|^2\mathrm{d}y_1 &\ll \sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\sum_{\substack{n=-N\\(*)}}^{N} e((n-y_2)\nu\tan\theta)\right|^2\\
&=\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\sum_{\substack{n=-N\\(*)}}^{N} e(n\nu\tan\theta)\right|^2,
\end{aligned}$$

so that

$$\begin{aligned}
\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}|G[\mathcal{P};\mathbf{y};r,\theta]|^2\mathrm{d}y_1\mathrm{d}y_2 &\ll \sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\sum_{\substack{n=-N\\(*)}}^{N} e(n\nu\tan\theta)\right|^2\\
&\ll \sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\min\{N^2,\|\nu\tan\theta\|^{-2}\},
\end{aligned}\tag{5}$$

where $\|\beta\|=\min_{n\in\mathbb{Z}}|\beta-n|$ for every $\beta\in\mathbb{R}$.

We need the following crucial estimate.

Lemma 1. *We have*

$$\int_0^{\pi/4}\left(\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\min\{N^2,\|\nu\tan\theta\|^{-2}\}\right)^{1/2}\mathrm{d}\theta \ll (\log N)^2.$$

Proof. Since $\tan\theta \asymp \theta$ if $0 \le \theta \le \pi/4$, it suffices to show that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \min\{N^2, \|n\omega\|^{-2}\}\right)^{1/2} \mathrm{d}\omega \ll (\log N)^2. \tag{6}$$

Clearly

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \min\{N^2, \|n\omega\|^{-2}\} \le \sum_{n=1}^{N^2} \frac{1}{n^2} \min\{N^2, \|n\omega\|^{-2}\} + 1,$$

so that

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \min\{N^2, \|n\omega\|^{-2}\}\right)^{1/2} \le \sum_{n=1}^{N^2} \frac{1}{n} \min\{N, \|n\omega\|^{-1}\} + 1. \tag{7}$$

Now, for every $n = 1, \ldots, N^2$, we have

$$\int_0^1 \min\{N, \|n\omega\|^{-1}\}\mathrm{d}\omega = 2n \int_0^{1/2n} \min\{N, (n\omega)^{-1}\}\mathrm{d}\omega \ll \log N. \tag{8}$$

Inequality (6) now follows on combining (7) and (8). ♣

By the Cauchy–Schwarz inequality, we have

$$\begin{aligned}\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} &|G[\mathcal{P};\mathbf{y};r,\theta]|\mathrm{d}y_1\mathrm{d}y_2 \\ &\ll \left(\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |G[\mathcal{P};\mathbf{y};r,\theta]|^2\mathrm{d}y_1\mathrm{d}y_2\right)^{1/2}.\end{aligned} \tag{9}$$

It follows from (4), (5), (9) and Lemma 1 that

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\int_0^{\pi/4}\int_1^{M(\theta)} |E[\mathcal{P};S(\mathbf{y};r,\theta)]|\mathrm{d}r\mathrm{d}\theta\mathrm{d}y_1\mathrm{d}y_2 \ll N(\log N)^2. \tag{10}$$

Note now that for every $\theta \in [0,\pi/4]$, every $r \ge 1$ and every $\mathbf{y} \in [-1/2,1/2]^2$, we have, writing $s = r + y_1\cos\theta + y_2\sin\theta$, that $|r-s| < 1$. It follows that since $S(\mathbf{y};r,\theta) = S(r + y_1\cos\theta + y_2\sin\theta,\theta)$, where $r + y_1\cos\theta + y_2\sin\theta \ge 0$, we must have

$$\int_2^{M(\theta)-1} |E[\mathcal{P};S(r,\theta)]|\mathrm{d}r \le \int_1^{M(\theta)} |E[\mathcal{P};S(\mathbf{y};r,\theta)]|\mathrm{d}r. \tag{11}$$

On the other hand,

$$\left(\int_0^2 + \int_{M(\theta)-1}^{M(\theta)}\right) |E[\mathcal{P};S(r,\theta)]|\mathrm{d}r \ll N. \tag{12}$$

It now follows from (10)–(12) that

$$\int_0^{\pi/4} \int_0^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r\mathrm{d}\theta \ll N(\log N)^2.$$

Similarly, for $j = 1, \ldots, 7$, we have

$$\int_{j\pi/4}^{(j+1)\pi/4} \int_0^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r\mathrm{d}\theta \ll N(\log N)^2.$$

Inequality (1) now follows.

3. A special case: U is a circular disc

Next, we consider the case when U is the closed disc of unit area and centred at the origin $\mathbf{0}$.

Let N be any given natural number. Again we consider a renormalized version of the problem, and take V to be the closed disc of area N and centred at the origin $\mathbf{0}$. However, if we simply attempt to take all the integer lattice points in V as our set $\mathcal{P}$, then the number of points of $\mathcal{P}$ can differ from N by an amount sufficiently large to make our task impossible (see Hardy [**4**] and pp. 183–308 of Landau [**5**]).

Our new idea is to introduce a set $\mathcal{P}$ such that the majority of points of $\mathcal{P}$ are integer lattice points in V, and that the remaining points give rise to a one–dimensional discrepancy along and near the boundary of V. More precisely, for any $\mathbf{x} = (x_1, x_2) \in \mathbb{Z}^2$, let

$$A(\mathbf{x}) = A(x_1, x_2) = [x_1 - 1/2, x_1 + 1/2] \times [x_2 - 1/2, x_2 + 1/2]; \tag{13}$$

in other words, $A(\mathbf{x})$ is the aligned closed square of unit area and centred at $\mathbf{x}$. Let

$$\mathcal{P}_1 = \{\mathbf{p} \in \mathbb{Z}^2 : A(\mathbf{p}) \subseteq V\}, \tag{14}$$

and write

$$V_1 = \bigcup_{\mathbf{p} \in \mathcal{P}_1} A(\mathbf{p}). \tag{15}$$

Note that the points of $\mathcal{P}_1$ form the majority of any point set $\mathcal{P}$ of N points in V. For the remaining points, let

$$V_2 = V \setminus V_1. \tag{16}$$

Then it is easy to see, writing $\pi M^2 = N$, that

$$\mu(V_2) \in \mathbb{N} \qquad \text{and} \qquad \mu(V_2) \ll M.$$

We partition V_2 as follows. Write

$$L = \mu(V_2), \tag{17}$$

and let

$$0 = \theta_0 < \theta_1 < \ldots < \theta_{L-1} < \theta_L = 1 \tag{18}$$

such that for every $j = 1, \ldots, L$, the set

$$R_j = \{\mathbf{x} \in V_2 : 2\pi\theta_{j-1} \le \arg \mathbf{x} < 2\pi\theta_j\} \tag{19}$$

satisfies

$$\mu(R_j) = 1. \tag{20}$$

For every $j = 1, \ldots, L$, let

$$\mathbf{p}_j \in R_j, \tag{21}$$

and write

$$\mathcal{P}_2 = \{\mathbf{p}_1, \ldots, \mathbf{p}_L\}. \tag{22}$$

If we now take

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2, \tag{23}$$

then clearly $\mathcal{P}$ contains exactly N points.

For every measurable set B in $\mathbb{R}^2$, let $Z[\mathcal{P}; B]$ denote the number of points of $\mathcal{P}$ in B, and write

$$E[\mathcal{P}; B] = Z[\mathcal{P}; B] - \mu(B \cap V).$$

Clearly, for every $j = 1, \ldots, L$, we have

$$E[\mathcal{P}; R_j] = 0. \tag{24}$$

We shall show that the set $\mathcal{P}$ satisfies

$$\int_0^{2\pi} \int_0^M |E[\mathcal{P}; S(r, \theta)]| \mathrm{d}r \mathrm{d}\theta \ll M (\log N)^2. \tag{25}$$

Again, suppose that $0 \le \theta \le \pi/4$.

As before, the line $T(r, \theta)$ is given by $x_1 \cos\theta + x_2 \sin\theta = r$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Furthermore, $T(r, \theta)$ intersects the boundary of V at the points

$$(r\cos\theta + (M^2 - r^2)^{1/2} \sin\theta, r\sin\theta - (M^2 - r^2)^{1/2} \cos\theta) \tag{26}$$

and

$$(r\cos\theta - (M^2 - r^2)^{1/2} \sin\theta, r\sin\theta + (M^2 - r^2)^{1/2} \cos\theta). \tag{27}$$

Let $T^{(1)}(r, \theta)$ denote the line segment joining the point $(r\cos\theta, r\sin\theta)$ and (26), and let $T^{(2)}(r, \theta)$ denote the line segment joining the point $(r\cos\theta, r\sin\theta)$ and (27).

Suppose first of all that $0 \le r \le M - 4$. Let

$$\begin{aligned} M^{(1)}(r, \theta) = \max\{n \in \mathbb{Z} : & \text{ there exists } m \in \mathbb{Z} \text{ such that } A(m, n) \cap V_2 \ne \emptyset \\ & \text{and } A(m, n) \cap T^{(1)}(r, \theta) \ne \emptyset\} \end{aligned}$$

and

$$M^{(2)}(r,\theta) = \min\{n \in \mathbb{Z} : \text{there exists } m \in \mathbb{Z} \text{ such that } A(m,n) \cap V_2 \neq \emptyset \\ \text{and } A(m,n) \cap T^{(2)}(r,\theta) \neq \emptyset\},$$

and let

$$I(r,\theta) = \{n \in \mathbb{Z} : M^{(1)}(r,\theta) < n < M^{(2)}(r,\theta)\}.$$

We can now write $S(r,\theta) \cap V$ as a union of subsets as follows. Let

$$S_0(r,\theta) = \bigcup_{\substack{\mathbf{x} \in \mathbb{Z}^2 \\ A(\mathbf{x}) \subseteq S(r,\theta) \cap V_1}} A(\mathbf{x}). \tag{28}$$

Also, let

$$S_1(r,\theta) = S(r,\theta) \cap \left(\bigcup_{n \in I(r,\theta)} \bigcup_{\substack{m \in \mathbb{Z} \\ A(m,n) \cap S(r,\theta) \neq \emptyset \\ A(m,n) \setminus S(r,\theta) \neq \emptyset}} A(m,n) \right) \tag{29}$$

(note here that the three conditions $n \in I(r,\theta)$, $A(m,n) \cap S(r,\theta) \neq \emptyset$ and $A(m,n) \setminus S(r,\theta) \neq \emptyset$ imply that we must have $A(m,n) \subseteq V_1$) and

$$S_2(r,\theta) = \bigcup_{\substack{j=1 \\ R_j \subseteq S(r,\theta)}}^{L} R_j. \tag{30}$$

The remainder of $S(r,\theta)$ consists of

$$W_1^{(1)}(r,\theta) = S(r,\theta) \cap V \cap \left(\bigcup_{n \leq M^{(1)}(r,\theta)} \bigcup_{\substack{m \in \mathbb{Z} \\ A(m,n) \subseteq V_1 \\ A(m,n) \cap S(r,\theta) \neq \emptyset \\ A(m,n) \setminus S(r,\theta) \neq \emptyset}} A(m,n) \right) \tag{31}$$

and

$$W_1^{(2)}(r,\theta) = S(r,\theta) \cap V \cap \left(\bigcup_{n \geq M^{(2)}(r,\theta)} \bigcup_{\substack{m \in \mathbb{Z} \\ A(m,n) \subseteq V_1 \\ A(m,n) \cap S(r,\theta) \neq \emptyset \\ A(m,n) \setminus S(r,\theta) \neq \emptyset}} A(m,n) \right), \tag{32}$$

as well as

$$W_2^{(1)}(r,\theta) = S(r,\theta) \cap \left(\bigcup_{\substack{j=1 \\ R_j \cap T^{(1)}(r,\theta) \neq \emptyset \\ R_j \setminus S(r,\theta) \neq \emptyset}}^{L} R_j \right) \tag{33}$$

and

$$W_2^{(2)}(r,\theta) = S(r,\theta) \cap \left(\bigcup_{\substack{j=1 \\ R_j \cap T^{(2)}(r,\theta) \neq \emptyset \\ R_j \setminus S(r,\theta) \neq \emptyset}}^{L} R_j \right). \tag{34}$$

It is not difficult to see that since $0 \leq r \leq M-4$, we have

$$S(r,\theta) \cap V = \left(\bigcup_{j=0}^{2} S_j(r,\theta) \right) \cup \left(\bigcup_{j=1}^{2} \bigcup_{k=1}^{2} W_j^{(k)}(r,\theta) \right).$$

Also, each pair B_1 and B_2 of the seven sets on the right-hand side satisfy $\mu(B_1 \cap B_2) = 0$ and $B_1 \cap B_2 \cap \mathcal{P} = \emptyset$. It follows that

$$E[\mathcal{P}; S(r,\theta)] = \sum_{j=0}^{2} E[\mathcal{P}; S_j(r,\theta)] + \sum_{j=1}^{2} \sum_{k=1}^{2} E[\mathcal{P}; W_j^{(k)}(r,\theta)]. \tag{35}$$

We shall estimate each of the terms on the right-hand side when $0 \leq r \leq M-4$.

Clearly

$$E[\mathcal{P}; S_0(r,\theta)] = 0, \tag{36}$$

as for each square $A(\mathbf{x})$ in $S_0(r,\theta)$, we have $Z[\mathcal{P}; A(\mathbf{x})] = \mu(A(\mathbf{x})) = 1$. Similarly

$$E[\mathcal{P}; S_2(r,\theta)] = 0 \tag{37}$$

in view of (24).

As before, let $\psi(z) = z - [z] - 1/2$ for every $z \in \mathbb{R}$.

Lemma 2. *Suppose that $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq M-4$. Then*

$$E[\mathcal{P}; S_1(r,\theta)] = - \sum_{n \in I(r,\theta)} \psi(n \tan\theta - r \sec\theta). \tag{38}$$

Proof. For each $n \in I(r,\theta)$, let

$$S_1(n,r,\theta) = S(r,\theta) \cap \left(\bigcup_{\substack{m \in \mathbb{Z} \\ A(m,n) \cap S(r,\theta) \neq \emptyset \\ A(m,n) \setminus S(r,\theta) \neq \emptyset}} A(m,n) \right).$$

Then

$$S_1(r,\theta) = \bigcup_{n \in I(r,\theta)} S_1(n,r,\theta).$$

Clearly

$$E[\mathcal{P}; S_1(r,\theta)] = \sum_{n \in I(r,\theta)} E[\mathcal{P}; S_1(n,r,\theta)]. \tag{39}$$

Now let $n \in I(r,\theta)$. Then there exists a greatest $m \in \mathbb{Z}$ such that $A(m,n) \cap S(r,\theta) \neq \emptyset$ and $A(m,n) \setminus S(r,\theta) \neq \emptyset$. It is not difficult to see that

$$Z[\mathcal{P}; S_1(n,r,\theta)] = [m + n\tan\theta - r\sec\theta + 1]$$

and

$$\mu(S_1(n,r,\theta)) = m + n\tan\theta - r\sec\theta + 1/2.$$

It follows that

$$E[\mathcal{P}; S_1(n,r,\theta)] = -\psi(n\tan\theta - r\sec\theta). \tag{40}$$

Clearly (38) follows on combining (39) and (40). ♣

Again, if we work out the Fourier expansion of the term $E[\mathcal{P}; S_1(r,\theta)]$, then the summation restriction $n \in I(r,\theta)$ prevents us from applying Parseval's theorem. As before, let $\mathbf{y} = (y_1, y_2) \in [-1/2, 1/2]^2$. For every $\theta \in [0, \pi/4]$ and every $r \geq 1$, define $T(\mathbf{y}; r, \theta)$ and $S(\mathbf{y}; r, \theta)$ as in (2) and (3). Note that $T(\mathbf{y}; r, \theta)$ intersects the boundary of V at the points

$$(s\cos\theta + (M^2 - s^2)^{1/2}\sin\theta, s\sin\theta - (M^2 - s^2)^{1/2}\cos\theta) \tag{41}$$

and

$$(s\cos\theta - (M^2 - s^2)^{1/2}\sin\theta, s\sin\theta + (M^2 - s^2)^{1/2}\cos\theta), \tag{42}$$

where $s = s(\mathbf{y}) = r + y_1\cos\theta + y_2\sin\theta$. Let $T^{(1)}(\mathbf{y}; r, \theta)$ denote the line segment joining the points $(s\cos\theta, s\sin\theta)$ and (41), and let $T^{(2)}(\mathbf{y}; r, \theta)$ denote the line segment joining the points $(s\cos\theta, s\sin\theta)$ and (42). For $1 \leq r \leq M - 4$, let

$$\begin{aligned} M^{(1)}(\mathbf{y}; r, \theta) = \max\{n \in \mathbb{Z} : \text{ there exists } m \in \mathbb{Z} \text{ such that } A(m,n) \cap V_2 \neq \emptyset \\ \text{and } A(m,n) \cap T^{(1)}(\mathbf{y}; r, \theta) \neq \emptyset\} \end{aligned}$$

and

$$\begin{aligned} M^{(2)}(\mathbf{y}; r, \theta) = \min\{n \in \mathbb{Z} : \text{ there exists } m \in \mathbb{Z} \text{ such that } A(m,n) \cap V_2 \neq \emptyset \\ \text{and } A(m,n) \cap T^{(2)}(\mathbf{y}; r, \theta) \neq \emptyset\}, \end{aligned}$$

and let

$$I(\mathbf{y}; r, \theta) = \{n \in \mathbb{Z} : M^{(1)}(\mathbf{y}; r, \theta) < n < M^{(2)}(\mathbf{y}; r, \theta)\}.$$

Now let

$$S_1(\mathbf{y};r,\theta)=S(\mathbf{y};r,\theta)\cap\left(\bigcup_{n\in I(\mathbf{y};r,\theta)}\bigcup_{\substack{m\in\mathbb{Z}\\A(m,n)\cap S(\mathbf{y};r,\theta)\neq\emptyset\\A(m,n)\setminus S(\mathbf{y};r,\theta)\neq\emptyset}}A(m,n)\right).$$

Then clearly

$$E[\mathcal{P};S_1(\mathbf{y};r,\theta)]=-\sum_{n\in I(\mathbf{y};r,\theta)}\psi(n\tan\theta-(r+y_1\cos\theta+y_2\sin\theta)\sec\theta).$$

We shall approximate $E[\mathcal{P};S_1(\mathbf{y};r,\theta)]$ by

$$G_1[\mathcal{P};\mathbf{y};r,\theta]=-\sum_{n\in I(r,\theta)}\psi(n\tan\theta-(r+y_1\cos\theta+y_2\sin\theta)\sec\theta).$$

Lemma 3. *For every $\mathbf{y}\in[-1/2,1/2]^2$, we have*

$$\int_0^{\pi/4}\int_1^{M-4}|E[\mathcal{P};S_1(\mathbf{y};r,\theta)]-G_1[\mathcal{P};\mathbf{y};r,\theta]|\mathrm{d}r\mathrm{d}\theta\ll M.$$

The proof of Lemma 3 will be given later, as the ideas are similar to those for studying the terms $E[\mathcal{P};W_j^{(k)}(r,\theta)]$.

Now $G_1[\mathcal{P};\mathbf{y};r,\theta]$ has the Fourier expansion

$$\begin{aligned}&\sum_{\nu\neq0}\frac{e(-(r+y_1\cos\theta+y_2\sin\theta)\nu\sec\theta)}{2\pi\mathrm{i}\nu}\sum_{n\in I(r,\theta)}e(n\nu\tan\theta)\\&\qquad=\sum_{\nu\neq0}\frac{e(-r\nu\sec\theta)}{2\pi\mathrm{i}\nu}\sum_{n\in I(r,\theta)}e((n-y_2)\nu\tan\theta)e(-y_1\nu).\end{aligned}\tag{43}$$

It follows that for every $y_2\in[-1/2,1/2]$, we have, by Parseval's theorem, that

$$\int_{-1/2}^{1/2}|G_1[\mathcal{P};\mathbf{y};r,\theta]|^2\mathrm{d}y_1\ll\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\left|\sum_{n\in I(r,\theta)}e(n\nu\tan\theta)\right|^2.$$

It follows that

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}|G_1[\mathcal{P};\mathbf{y};r,\theta]|^2\mathrm{d}y_1\mathrm{d}y_2\ll\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\min\{M^2,\|\nu\tan\theta\|^{-2}\},$$

so that by the Cauchy–Schwarz inequality,

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} |G_1[\mathcal{P};\mathbf{y};r,\theta]|\,\mathrm{d}y_1\mathrm{d}y_2 \ll \left(\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\min\{M^2,\|\nu\tan\theta\|^{-2}\}\right)^{1/2}. \tag{44}$$

To study the terms $E[\mathcal{P};W_j^{(k)}(r,\theta)]$, we have

Lemma 4. *For $j,k\in\{1,2\}$, we have*

$$\int_0^{\pi/4}\int_0^{M-4}|E[\mathcal{P};W_j^{(k)}(r,\theta)]|\mathrm{d}r\mathrm{d}\theta \ll M.$$

Suppose that $0\le\theta\le\pi/4$ and $0\le r\le M-4$. Let

$$I^{(1)}(r,\theta)=\{n\in\mathbb{Z}: r\sin\theta-(M^2-r^2)^{1/2}\cos\theta\le n\le M^{(1)}(r,\theta)\}$$

and

$$I^{(2)}(r,\theta)=\{n\in\mathbb{Z}: M^{(2)}(r,\theta)\le n\le r\sin\theta+(M^2-r^2)^{1/2}\cos\theta\}.$$

Note that $r\sin\theta\pm(M^2-r^2)^{1/2}\cos\theta$ are the second coordinates of the two points of intersection of $T(r,\theta)$ and the boundary of V, and that

$$\begin{aligned}&I(r,\theta)\cup I^{(1)}(r,\theta)\cup I^{(2)}(r,\theta)\\&\quad=\{n\in\mathbb{Z}: r\sin\theta-(M^2-r^2)^{1/2}\cos\theta\le n\le r\sin\theta+(M^2-r^2)^{1/2}\cos\theta\}.\end{aligned}$$

Furthermore, the three sets on the left–hand side are pairwise disjoint.

If $0\le\theta\le\pi/4$ and $0\le r\le M-4$, it is not difficult to see that for every $j,k\in\{1,2\}$, we have

$$|E[\mathcal{P};W_j^{(k)}(r,\theta)]|\ll\operatorname{card}\left(I^{(k)}(r,\theta)\right).$$

Lemma 4 will follow if we can prove

Lemma 5. *For $j,k\in\{1,2\}$, we have*

$$\int_0^{\pi/4}\int_0^{M-4}\operatorname{card}\left(I^{(k)}(r,\theta)\right)\mathrm{d}r\mathrm{d}\theta\ll M.$$

To prove Lemma 3, note that if $0\le\theta\le\pi/4$, $1\le r\le M-4$ and $\mathbf{y}\in[-1/2,1/2]^2$, we have

$$E[\mathcal{P};S_1(\mathbf{y};r,\theta)]-G_1[\mathcal{P};\mathbf{y};r,\theta]\ll\min\{M,\operatorname{card}(I(r,\theta)\triangle I(\mathbf{y};r,\theta))\}, \tag{45}$$

where $B_1 \triangle B_2$ denotes the symmetric difference between the sets B_1 and B_2. Clearly $I(\mathbf{y}; r, \theta) = I(s, \theta)$, where $s = r + y_1 \cos\theta + y_2 \sin\theta \geq 0$. In this case,

$$I(r,\theta) \triangle I(s,\theta) \subseteq \bigcup_{k=1}^{2} \left(I^{(k)}(r,\theta) \cup I^{(k)}(s,\theta) \right). \tag{46}$$

Note now that $|r - s| < 1$, so it follows from (45), (46) and Lemma 5 that

$$\int_0^{\pi/4} \int_2^{M-5} |E[\mathcal{P}; S_1(\mathbf{y}; r,\theta)] - G_1[\mathcal{P}; \mathbf{y}; r, \theta]| \mathrm{d}r \mathrm{d}\theta$$

$$\ll \sum_{k=1}^{2} \int_0^{\pi/4} \int_1^{M-4} \mathrm{card}\left(I^{(k)}(r,\theta) \right) \mathrm{d}r \mathrm{d}\theta \ll M.$$

Lemma 3 now follows on combining this and the simple observation that

$$\int_0^{\pi/4} \left(\int_1^2 + \int_{M-5}^{M-4} \right) |E[\mathcal{P}; S_1(\mathbf{y}; r,\theta)] - G_1[\mathcal{P}; \mathbf{y}; r, \theta]| \mathrm{d}r \mathrm{d}\theta \ll M.$$

Proof of Lemma 5. Note that $T^{(1)}(r,\theta)$ intersects ∂V, the boundary of V, at the point (26). Clearly $n + 1 \notin I^{(1)}(r,\theta)$ if the distance between the points

$$(-n\tan\theta + r\sec\theta, n) \in T^{(1)}(r,\theta) \qquad \text{and} \qquad \left((M^2 - n^2)^{1/2}, n \right) \in \partial V$$

exceeds 1. It follows that

$$\mathrm{card}\left(I^{(1)}(r,\theta) \right) \ll 1 + v - r\sin\theta + (M^2 - r^2)^{1/2} \cos\theta, \tag{47}$$

where

$$(M^2 - v^2)^{1/2} + v\tan\theta - r\sec\theta = 1 \tag{48}$$

and $v < r\sin\theta$. Elementary calculation gives

$$v = \sin\theta\cos\theta + r\sin\theta - (M^2 - (r + \cos\theta)^2)^{1/2} \cos\theta, \tag{49}$$

so that, combining (47) and (49), we have

$$\begin{aligned} &\mathrm{card}\left(I^{(1)}(r,\theta) \right) \\ &\quad \ll 1 + \sin\theta\cos\theta + (M^2 - r^2)^{1/2} \cos\theta - (M^2 - (r + \cos\theta)^2)^{1/2} \cos\theta \\ &\quad \ll 1 + \frac{r}{(M^2 - r^2)^{1/2}} \end{aligned}$$

since $r \leq M - 4$. Clearly

$$\int_0^{\pi/4} \int_0^{M-4} \mathrm{card}\left(I^{(1)}(r,\theta)\right) \mathrm{d}r\mathrm{d}\theta \ll M.$$

On the other hand, $T^{(2)}(r,\theta)$ intersects ∂V at the point (27). Suppose first of all that $r \geq M \sin\theta$, so that $r\cos\theta - (M^2 - r^2)^{1/2}\sin\theta \geq 0$. Then $n - 1 \notin I^{(2)}(r,\theta)$ if the distance between the points

$$(-n\tan\theta + r\sec\theta, n) \in T^{(2)}(r,\theta) \qquad \text{and} \qquad \left((M^2 - n^2)^{1/2}, n\right) \in \partial V \tag{50}$$

exceeds 1. It follows that if $r \geq M \sin\theta$, then

$$\mathrm{card}\left(I^{(2)}(r,\theta)\right) \ll 1 + r\sin\theta + (M^2 - r^2)^{1/2}\cos\theta - v, \tag{51}$$

where v satisfies (48) and

$$v > r\sin\theta. \tag{52}$$

Elementary calculation gives

$$v = \sin\theta\cos\theta + r\sin\theta + \left(M^2 - (r + \cos\theta)^2\right)^{1/2}\cos\theta, \tag{53}$$

so that, combining (51) and (53), we have

$$\begin{aligned}
&\mathrm{card}\left(I^{(2)}(r,\theta)\right) \\
&\qquad \ll 1 - \sin\theta\cos\theta + (M^2 - r^2)^{1/2}\cos\theta - \left(M^2 - (r + \cos\theta)^2\right)^{1/2}\cos\theta \\
&\qquad \ll 1 + \frac{r}{(M^2 - r^2)^{1/2}}
\end{aligned}$$

since $r \leq M - 4$. Suppose now that $r < M \sin\theta$, so that $r\cos\theta - (M^2 - r^2)^{1/2}\sin\theta < 0$. Then $n - 1 \notin I^{(2)}(r,\theta)$ if the distance between the points (50) and the distance between the points

$$(-n\tan\theta + r\sec\theta, n) \in T^{(2)}(r,\theta) \qquad \text{and} \qquad \left(-(M^2 - n^2)^{1/2}, n\right) \in \partial V$$

exceeds 1. It follows that (51) must hold both when v satisfies (48) and (52) and when v satisfies

$$(M^2 - v^2)^{1/2} - v\tan\theta + r\sec\theta = 1$$

and (52). Clearly we only need to investigate the latter case. Elementary calculation gives

$$v = r\sin\theta - \sin\theta\cos\theta + \left(M^2 - (r - \cos\theta)^2\right)^{1/2}\cos\theta, \tag{54}$$

so that, combining (51) and (54), we have

$$\begin{aligned}
&\operatorname{card}\left(I^{(2)}(r,\theta)\right)\\
&\qquad \ll 1+\sin\theta\cos\theta+\left(M^2-r^2\right)^{1/2}\cos\theta-\left(M^2-(r-\cos\theta)^2\right)^{1/2}\cos\theta\\
&\qquad \ll 1+\frac{r}{\left(M^2-r^2\right)^{1/2}}
\end{aligned}$$

since $r\leq M-4$. Clearly

$$\int_0^{\pi/4}\int_0^{M-4}\operatorname{card}\left(I^{(2)}(r,\theta)\right)\mathrm{d}r\mathrm{d}\theta\ll M. \clubsuit$$

It now follows from (35)–(37) and Lemma 4 that

$$\int_0^{\pi/4}\int_0^{M-4}|E[\mathcal{P};S(r,\theta)]|\mathrm{d}r\mathrm{d}\theta\ll M+\int_0^{\pi/4}\int_0^{M-4}|E[\mathcal{P};S_1(r,\theta)]|\mathrm{d}r\mathrm{d}\theta. \tag{55}$$

On the other hand, it is easy to see that if $0\leq\theta\leq\pi/4$ and $M-4\leq r\leq M$, we have

$$Z[\mathcal{P};S(r,\theta)]=Z[\mathcal{P};S(r,\theta)\cap V_1]+Z[\mathcal{P};S(r,\theta)\cap V_2]\ll M+L\ll M$$

and $\mu(S(r,\theta)\cap V)\ll M$, so that $E[\mathcal{P};S(r,\theta)]\ll M$, whence

$$\int_0^{\pi/4}\int_{M-4}^{M}|E[\mathcal{P};S(r,\theta)]|\mathrm{d}r\mathrm{d}\theta\ll M. \tag{56}$$

Combining (55) and (56), we have

$$\int_0^{\pi/4}\int_0^{M}|E[\mathcal{P};S(r,\theta)]|\mathrm{d}r\mathrm{d}\theta\ll M+\int_0^{\pi/4}\int_0^{M-4}|E[\mathcal{P};S_1(r,\theta)]|\mathrm{d}r\mathrm{d}\theta. \tag{57}$$

Combining Lemma 1, (44) and Lemma 3, we have

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}\int_0^{\pi/4}\int_1^{M-4}|E[\mathcal{P};S_1(\mathbf{y};r,\theta)]|\mathrm{d}r\mathrm{d}\theta\mathrm{d}y_1\mathrm{d}y_2\ll M(\log N)^2. \tag{58}$$

Note again that for every $\theta\in[0,\pi/4]$, every $r\geq 1$ and every $\mathbf{y}\in[-1/2,1/2]^2$, we have, writing $s=r+y_1\cos\theta+y_2\sin\theta$, that $|r-s|<1$. It follows that since $S_1(\mathbf{y};r,\theta)=S_1(r+y_1\cos\theta+y_2\sin\theta,\theta)$, where $r+y_1\cos\theta+y_2\sin\theta\geq 0$, we must have

$$\int_2^{M-5}|E[\mathcal{P};S_1(r,\theta)]|\mathrm{d}r\leq\int_1^{M-4}|E[\mathcal{P};S_1(\mathbf{y};r,\theta)]|\mathrm{d}r. \tag{59}$$

On the other hand, $|E[\mathcal{P}; S_1(r,\theta)]| \ll M$ always, so that

$$\left(\int_0^2 + \int_{M-5}^{M-4}\right) |E[\mathcal{P}; S_1(r,\theta)]| \mathrm{d}r \ll M. \tag{60}$$

It now follows from (58)–(60) that

$$\int_0^{\pi/4} \int_0^{M-4} |E[\mathcal{P}; S_1(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll M(\log N)^2. \tag{61}$$

Combining (57) and (61), we have

$$\int_0^{\pi/4} \int_0^{M} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll M(\log N)^2.$$

Similarly, for $j = 1, \ldots, 7$, we have

$$\int_{j\pi/4}^{(j+1)\pi/4} \int_0^{M} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll M(\log N)^2.$$

Inequality (25) now follows.

4. Proof of the Main Theorem

Finally, we consider the problem in general, where U is a closed convex set in $\mathbb{R}^2$, and with centre of gravity at $\mathbf{0}$.

Let N be any given natural number. As in the special cases considered earlier, we let $V = \{N^{1/2}\mathbf{x} : \mathbf{x} \in U\}$, so that $\mu(V) = N$. Our approach is similar to that when V is a circular disc. Indeed, we define $\mathcal{P}$, $\mathcal{P}_1$, $\mathcal{P}_2$, V_1, V_2, L and R_j $(j = 1, \ldots, L)$ in terms of V by (13)–(23), and note that

$$\mu(V_2) \in \mathbb{Z} \qquad \text{and} \qquad \mu(V_2) \ll N^{1/2}.$$

For every measurable set B in $\mathbb{R}^2$, let $Z[\mathcal{P}; B]$ denote the number of points of $\mathcal{P}$ in B, and write

$$E[\mathcal{P}; B] = Z[\mathcal{P}; B] - \mu(B \cap V).$$

Then (24) holds for every $j = 1, \ldots, L$. We shall show that the set $\mathcal{P}$ satisfies

$$\int_0^{2\pi} \int_0^{M(\theta)} |E[\mathcal{P}; B]| \mathrm{d}r \mathrm{d}\theta \ll N^{1/2}(\log N)^2, \tag{62}$$

where, for every $\theta \in [0, 2\pi]$, we have $M(\theta) = N^{1/2}R(\theta)$.

Again, suppose that $0 \le \theta \le \pi/4$.

As before, the line $T(r,\theta)$ is given by $x_1 \cos\theta + x_2 \sin\theta = r$, where $\mathbf{x} = (x_1, x_2) \in \mathbb{R}^2$. Furthermore, $T(r,\theta)$ intersects the boundary of V at the points

$$\mathbf{u}^{(1)}(r,\theta) = \left(u_1^{(1)}(r,\theta), u_2^{(1)}(r,\theta)\right) \tag{63}$$

and

$$\mathbf{u}^{(2)}(r,\theta) = \left(u_1^{(2)}(r,\theta), u_2^{(2)}(r,\theta)\right), \tag{64}$$

with the restriction that $u_2^{(1)}(r,\theta) \le u_2^{(2)}(r,\theta)$. Here the argument is slightly more complicated than before. Consider the line segment $T(r,\theta) \cap V$. We need the following geometric lemma. To state this, we need some notation. Let

$$R_{\max} = \sup\{M(\theta) : 0 \le \theta < 2\pi\} \qquad \text{and} \qquad R_{\min} = \inf\{M(\theta) : 0 \le \theta < 2\pi\},$$

and let

$$\rho = \frac{R_{\min}}{R_{\max}}.$$

Also, for $\theta \in [0, \pi/4]$ and $r \in [0, M(\theta)]$, let

$$l(r,\theta) = \left|\mathbf{u}^{(1)}(r,\theta) - \mathbf{u}^{(2)}(r,\theta)\right|;$$

in other words, $l(r,\theta)$ is the length of the line segment $T(r,\theta) \cap V$. Furthermore, let $\mathbf{p}(r,\theta)$ denote the midpoint of $T(r,\theta) \cap V$.

We shall assume that N is sufficiently large.

Lemma 6. *Suppose that $\theta \in [0, \pi/4]$. Suppose further that $0 \le r \le M(\theta) - 48$ and $l(r,\theta) \ge 96/\rho$. Then the square of side 12, centred at $\mathbf{p}(r,\theta)$ and with one side parallel to $T(r,\theta)$, lies in V.*

Proof. It clearly suffices to show that the sets $S(r,\theta) \cap V$ and $(V \setminus S(r,\theta)) \cup (T(r,\theta) \cap V)$ each contains a rectangle of sides 6 and 12, with the point $\mathbf{p}(r,\theta)$ as the midpoint of one of the long sides.

(I) We shall first consider $S(r,\theta) \cap V$. Let $\mathbf{v}(r,\theta) \in S(r,\theta) \cap V$ be of maximal (perpendicular) distance from the line $T(r,\theta)$. By a suitable translation and rotation, we may assume that $\mathbf{p}(r,\theta)$ is the point $(0,0)$, that $\mathbf{u}^{(1)}(r,\theta)$ and $\mathbf{u}^{(2)}(r,\theta)$ are the points $(y,0)$ and $(-y,0)$ respectively, where $2y = l(r,\theta)$, and that $\mathbf{v}(r,\theta)$ is the point (u,x), where $x \ge 0$ (the reader is advised to draw a picture). We may further assume, without loss of generality, that $u \ge 0$. Clearly, it suffices to show, in view of the convexity of V, that the rectangle with vertices $(\pm 6, 0)$ and $(\pm 6, 6)$ is contained in the triangle with vertices $(\pm y, 0)$ and (u,x). In view of our assumption $u \ge 0$, it suffices to show that if $x \ge 48$ and $y \ge 48/\rho$, then

$$\frac{x}{y+u} \ge \frac{6}{y-6},$$

i.e. $x(y-6) \geq 6u+6y$. Now if $u \leq y$, then since $x \geq 24$ and $y \geq 12$, we have

$$x(y-6) \geq \frac{xy}{2} \geq 12y \geq 6u+6y.$$

On the other hand, if $u > y$, then by the convexity of V, we must have $x \geq R_{\min}$ and $u-y \leq R_{\max}$, so that

$$\frac{x}{u-y} \geq \frac{R_{\min}}{R_{\max}} = \rho,$$

i.e. $x+\rho y \geq \rho u$. Since $x \geq 48$ and $y \geq 48/\rho$, we have, noting that $\rho \leq 1$, that

$$x(y-6) \geq \frac{xy}{2} \geq \frac{12x}{\rho} + 12y = \frac{12}{\rho}(x+\rho y) \geq 12u > 6u+6y.$$

(II) We now consider $(V \setminus S(r,\theta)) \cup (T(r,\theta) \cap V)$. Let $\mathbf{u}^{(1)}(0,\theta)$ and $\mathbf{u}^{(2)}(0,\theta)$ denote the endpoints of the line segment $T(0,\theta) \cap V$. By a suitable rotation about the centre $\mathbf{0}$ of V, we may assume that $T(r,\theta)$ is a horizontal line (the reader is advised to draw a picture), that the point $\mathbf{p}(r,\theta)$ is denoted by (u,r), that the points $\mathbf{u}^{(1)}(r,\theta)$ and $\mathbf{u}^{(2)}(r,\theta)$ are denoted by $(u+y,r)$ and $(u-y,r)$ respectively, where $2y = l(r,\theta)$. Clearly, in view of convexity, it suffices to show that if $y \geq 48/\rho$, then the points $(u \pm 6, r-6)$ are contained in the triangle with vertices $(0,0)$ and $(u \pm y, r)$. Again, in view of convexity, it suffices to show that if $y \geq 48/\rho$, then

$$\frac{6}{y-6} \leq \frac{R_{\min}}{R_{\max}} = \rho,$$

i.e. $y \geq 6(1+\rho)/\rho$. This last inequality clearly holds if $y \geq 48/\rho$, since $\rho \leq 1$. ♣

For every $\theta \in [0,\pi/4]$ and $r \in [0, M(\theta)]$, let $SQ(r,\theta)$ denote the square of side 12, centred at $\mathbf{p}(r,\theta)$ and with one side parallel to $T(r,\theta)$. Further, let

$$M^*(\theta) = \sup\{0 \leq r \leq M(\theta) : SQ(r,\theta) \subseteq V\}.$$

Then clearly

Lemma 7. *For every $\theta \in [0,\pi/4]$, either*

$$M^*(\theta) \geq M(\theta) - 48$$

or

$$l\left(M^*(\theta),\theta\right) \leq \frac{96}{\rho}.$$

We shall also need

Lemma 8. *For every $\theta \in [0, \pi/4]$ and every $r \in [M^*(\theta), M(\theta)]$, we have*

$$l(r, \theta) \leq 2l\left(M^*(\theta), \theta\right)$$

if N is sufficiently large.

Proof. We may assume that N is sufficiently large so that $l(0, \theta) \geq 96/\rho$. Suppose first of all that $l(M^*(\theta), \theta) \leq l(0, \theta)$. Then in view of convexity, we must have $l(r, \theta) \leq l\left(M^*(\theta), \theta\right)$ if $r \geq M^*(\theta)$. Suppose now that $l(M^*(\theta), \theta) > l(0, \theta)$. Then, again by convexity, we must have

$$\frac{l(r, \theta)}{r} \leq \frac{l(M^*(\theta), \theta)}{M^*(\theta)}.$$

In view of Lemma 7 and our assumption that $l(0, \theta) \geq 96/\rho$, we must have $M^*(\theta) \geq M(\theta) - 48$. It follows that

$$l(r, \theta) \leq \frac{r}{M(\theta) - 48} l\left(M^*(\theta), \theta\right) \leq \frac{M(\theta)}{M(\theta) - 48} l\left(M^*(\theta), \theta\right).$$

Note now that $M(\theta) \to \infty$ as $N \to \infty$. ♣

For every $\theta \in [0, \pi/4]$ and every $r \in [0, M(\theta)]$, let $T^{(1)}(r, \theta)$ denote the line segment joining the points $\mathbf{p}(r, \theta)$ and $\mathbf{u}^{(1)}(r, \theta)$, and let $T^{(2)}(r, \theta)$ denote the line segment joining the points $\mathbf{p}(r, \theta)$ and $\mathbf{u}^{(2)}(r, \theta)$.

Suppose first of all that $0 \leq r \leq M^*(\theta)$. As before, let

$$\begin{aligned} M^{(1)}(r, \theta) = \max\{n \in \mathbb{Z} : \text{there exists } m \in \mathbb{Z} \text{ such that } A(m, n) \cap V_2 \neq \emptyset \\ \text{and } A(m, n) \cap T^{(1)}(r, \theta) \neq \emptyset\} \end{aligned} \tag{65}$$

and

$$\begin{aligned} M^{(2)}(r, \theta) = \min\{n \in \mathbb{Z} : \text{there exists } m \in \mathbb{Z} \text{ such that } A(m, n) \cap V_2 \neq \emptyset \\ \text{and } A(m, n) \cap T^{(2)}(r, \theta) \neq \emptyset\}, \end{aligned} \tag{66}$$

and let

$$I(r, \theta) = \{n \in \mathbb{Z} : M^{(1)}(r, \theta) < n < M^{(2)}(r, \theta)\}. \tag{67}$$

Then in view of Lemma 6, we must have $I(r, \theta) \neq \emptyset$. We now write $S(r, \theta) \cap V$ in the form

$$S(r, \theta) \cap V = \left(\bigcup_{j=0}^{2} S_j(r, \theta) \right) \cup \left(\bigcup_{j=1}^{2} \bigcup_{k=1}^{2} W_j^{(k)}(r, \theta) \right),$$

where the seven sets on the right-hand side are defined by (28)–(34). Clearly, each pair B_1 and B_2 of these seven sets satisfy $\mu(B_1 \cap B_2) = 0$ and $B_1 \cap B_2 \cap \mathcal{P} = \emptyset$. It follows that

$$E[\mathcal{P}; S(r, \theta)] = \sum_{j=0}^{2} E[\mathcal{P}; S_j(r, \theta)] + \sum_{j=1}^{2} \sum_{k=1}^{2} E[\mathcal{P}; W_j^{(k)}(r, \theta)]. \tag{68}$$

We shall estimate each of the terms on the right-hand side when $0 \leq r \leq M^*(\theta)$.

Clearly

$$E[\mathcal{P}; S_0(r,\theta)] = E[\mathcal{P}; S_2(r,\theta)] = 0 \tag{69}$$

as before. Also, as in Lemma 2, we have, writing $\psi(z) = z - [z] - 1/2$ for every $z \in \mathbb{R}$, that

Lemma 9. *Suppose that $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq M^*(\theta)$. Then*

$$E[\mathcal{P}; S_1(r,\theta)] = - \sum_{n \in I(r,\theta)} \psi(n \tan\theta - r\sec\theta).$$

As in the special case of the disc, the summation restriction $n \in I(r,\theta)$ again prevents us from applying Parseval's theorem to the Fourier expansion of the term $E[\mathcal{P}; S_1(r,\theta)]$. We can overcome this in a similar way as before. Let $\mathbf{y} = (y_1, y_2) \in [-1/2, 1/2]^2$. For every $\theta \in [0, \pi/4]$ and every $r \geq 1$, define $T(\mathbf{y}; r, \theta)$ and $S(\mathbf{y}; r, \theta)$ as in (2) and (3). Suppose now that $T(\mathbf{y}; r, \theta)$ intersects the boundary of V at the points

$$\mathbf{u}^{(1)}(\mathbf{y}; r, \theta) = \left(u_1^{(1)}(\mathbf{y}; r, \theta), u_2^{(1)}(\mathbf{y}; r, \theta)\right)$$

and

$$\mathbf{u}^{(2)}(\mathbf{y}; r, \theta) = \left(u_1^{(2)}(\mathbf{y}; r, \theta), u_2^{(2)}(\mathbf{y}; r, \theta)\right),$$

with the restriction that $u_2^{(1)}(\mathbf{y}; r, \theta) \leq u_2^{(2)}(\mathbf{y}; r, \theta)$. Let $T^{(1)}(\mathbf{y}; r, \theta)$ denote the line segment joining the points $\mathbf{p}(\mathbf{y}; r, \theta)$ and $\mathbf{u}^{(1)}(\mathbf{y}; r, \theta)$, and let $T^{(2)}(\mathbf{y}; r, \theta)$ denote the line segment joining the points $\mathbf{p}(\mathbf{y}; r, \theta)$ and $\mathbf{u}^{(2)}(\mathbf{y}; r, \theta)$, where $\mathbf{p}(\mathbf{y}; r, \theta)$ denotes the midpoint of the line segment $T(\mathbf{y}; r, \theta) \cap V$. For $1 \leq r \leq M^*(\theta)$, let

$$\begin{aligned} M^{(1)}(\mathbf{y}; r, \theta) = \max\{n \in \mathbb{Z} : \text{there exists } m \in \mathbb{Z} \text{ such that } A(m,n) \cap V_2 \neq \emptyset \\ \text{and } A(m,n) \cap T^{(1)}(\mathbf{y}; r, \theta) \neq \emptyset\} \end{aligned}$$

and

$$\begin{aligned} M^{(2)}(\mathbf{y}; r, \theta) = \min\{n \in \mathbb{Z} : \text{there exists } m \in \mathbb{Z} \text{ such that } A(m,n) \cap V_2 \neq \emptyset \\ \text{and } A(m,n) \cap T^{(2)}(\mathbf{y}; r, \theta) \neq \emptyset\}, \end{aligned}$$

and let

$$I(\mathbf{y}; r, \theta) = \{n \in \mathbb{Z} : M^{(1)}(\mathbf{y}; r, \theta) < n < M^{(2)}(\mathbf{y}; r, \theta)\}.$$

Now let, as before,

$$S_1(\mathbf{y}; r, \theta) = S(\mathbf{y}; r, \theta) \cap \left(\bigcup_{n \in I(\mathbf{y}; r, \theta)} \bigcup_{\substack{m \in \mathbb{Z} \\ A(m,n) \cap S(\mathbf{y}; r, \theta) \neq \emptyset \\ A(m,n) \setminus S(\mathbf{y}; r, \theta) \neq \emptyset}} A(m,n) \right).$$

Then clearly

$$E[\mathcal{P};S_1(\mathbf{y};r,\theta)]=-\sum_{n\in I(\mathbf{y};r,\theta)}\psi(n\tan\theta-(r+y_1\cos\theta+y_2\sin\theta)\sec\theta).$$

We shall approximate $E[\mathcal{P};S_1(\mathbf{y};r,\theta)]$ by

$$G_1[\mathcal{P};\mathbf{y};r,\theta]=-\sum_{n\in I(r,\theta)}\psi(n\tan\theta-(r+y_1\cos\theta+y_2\sin\theta)\sec\theta).$$

Corresponding to Lemma 3, we have

Lemma 10. *For every* $\mathbf{y}\in[-1/2,1/2]^2$, *we have*

$$\int_0^{\pi/4}\int_1^{M^*(\theta)}|E[\mathcal{P};S_1(\mathbf{y};r,\theta)]-G_1[\mathcal{P};\mathbf{y};r,\theta]|\mathrm{d}r\mathrm{d}\theta\ll N^{1/2}.$$

We shall prove Lemma 10 later.

Now $G_1[\mathcal{P};\mathbf{y};r,\theta]$ has the Fourier expansion (43). It follows, as before, that

$$\int_{-1/2}^{1/2}\int_{-1/2}^{1/2}|G_1[\mathcal{P};\mathbf{y};r,\theta]|\,\mathrm{d}y_1\mathrm{d}y_2\ll\left(\sum_{\nu=1}^{\infty}\frac{1}{\nu^2}\min\{N,\|\nu\tan\theta\|^{-2}\}\right)^{1/2}\tag{70}$$

for every $\theta\in[0,\pi/4]$ and $r\in[0,M^*(\theta)]$.

To study the terms $E[\mathcal{P};W_j^{(k)}(r,\theta)]$, we have the following analogue of Lemma 4.

Lemma 11. *For* $j,k\in\{1,2\}$, *we have*

$$\int_0^{\pi/4}\int_0^{M^*(\theta)}|E[\mathcal{P};W_j^{(k)}(r,\theta)]|\mathrm{d}r\mathrm{d}\theta\ll N^{1/2}.$$

Suppose that $0\le\theta\le\pi/4$ and $0\le r\le M^*(\theta)$. Let

$$I^{(1)}(r,\theta)=\{n\in\mathbb{Z}:u_2^{(1)}(r,\theta)\le n\le M^{(1)}(r,\theta)\}\tag{71}$$

and

$$I^{(2)}(r,\theta)=\{n\in\mathbb{Z}:M^{(2)}(r,\theta)\le n\le u_2^{(2)}(r,\theta)\}.$$

Then clearly

$$I(r,\theta)\cup I^{(1)}(r,\theta)\cup I^{(2)}(r,\theta)=\{n\in\mathbb{Z}:u_2^{(1)}(r,\theta)\le n\le u_2^{(2)}(r,\theta)\}.$$

Furthermore, the three sets on the left–hand side are pairwise disjoint.

If $0 \leq \theta \leq \pi/4$ and $0 \leq r \leq M^*(\theta)$, it is not difficult to see that for every $j,k \in \{1,2\}$, we have

$$|E[\mathcal{P}; W_j^{(k)}(r,\theta)]| \ll \mathrm{card}\left(I^{(k)}(r,\theta)\right).$$

Lemma 11 will follow from the analogue of Lemma 5 below.

Lemma 12. *For $j,k \in \{1,2\}$, we have*

$$\int_0^{\pi/4}\int_0^{M^*(\theta)} \mathrm{card}\left(I^{(k)}(r,\theta)\right) \mathrm{d}r\mathrm{d}\theta \ll N^{1/2}.$$

To prove Lemma 10, note that if $0 \leq \theta \leq \pi/4$, $1 \leq r \leq M^*(\theta)$ and $\mathbf{y} \in [-1/2,1/2]^2$, we have

$$E[\mathcal{P}; S_1(\mathbf{y};r,\theta)] - G_1[\mathcal{P};\mathbf{y};r,\theta] \ll \min\left\{N^{1/2}, \mathrm{card}(I(r,\theta)\triangle I(\mathbf{y};r,\theta))\right\}, \qquad (72)$$

where $B_1 \triangle B_2$ denotes the symmetric difference between the sets B_1 and B_2. Clearly $I(\mathbf{y};r,\theta) = I(s,\theta)$, where $s = r + y_1\cos\theta + y_2\sin\theta \geq 0$. In this case,

$$I(r,\theta)\triangle I(s,\theta) \subseteq \bigcup_{k=1}^{2}\left(I^{(k)}(r,\theta) \cup I^{(k)}(s,\theta)\right). \qquad (73)$$

Note now that $|r-s| < 1$, so it follows from (72), (73) and Lemma 12 that

$$\int_0^{\pi/4}\int_2^{M^*(\theta)-1} |E[\mathcal{P}; S_1(\mathbf{y};r,\theta)] - G_1[\mathcal{P};\mathbf{y};r,\theta]| \mathrm{d}r\mathrm{d}\theta$$
$$\ll \sum_{k=1}^{2}\int_0^{\pi/4}\int_1^{M^*(\theta)} \mathrm{card}\left(I^{(k)}(r,\theta)\right)\mathrm{d}r\mathrm{d}\theta \ll N^{1/2}.$$

Lemma 10 now follows on combining this and the simple observation that

$$\int_0^{\pi/4}\left(\int_1^2 + \int_{M^*(\theta)-1}^{M^*(\theta)}\right)|E[\mathcal{P}; S_1(\mathbf{y};r,\theta)] - G_1[\mathcal{P};\mathbf{y};r,\theta]| \mathrm{d}r\mathrm{d}\theta \ll N^{1/2}.$$

Note that Lemma 12 is a generalization of Lemma 5. Our proof, however, is necessarily different. In our earlier proof of Lemma 5, we use explicitly the equation of ∂V, the boundary of V. In the general case, such information is clearly not available.

Our proof here is based on the following simple geometric observation.

Consider the points $\mathbf{u}^{(1)}(n,\theta)$, where $n \in \mathbb{Z}$ and $0 \leq n \leq M(\theta)$. We extend this definition in the natural way to $n = -1,-2,\ldots,-6$. For each $\theta \in [0,\pi/4]$ and

$n = -6, \ldots, -1, 0, 1, \ldots, [M(\theta)]$, let $\mathcal{N}_\theta(n)$ denote the area of the rectangle with one edge on $T(n, \theta)$ and with vertices $\mathbf{u}^{(1)}(n, \theta)$ and $\mathbf{u}^{(1)}(n+1, \theta)$.

Lemma 13. *Suppose that* $0 \le \theta \le \pi/4$ *and* $0 \le r \le M^*(\theta)$. *Then*

$$I^{(1)}(r, \theta) \le \max\left\{6, \sum_{i=1}^{6} \mathcal{N}_\theta(n+i, \theta), \sum_{i=1}^{6} \mathcal{N}_\theta(n-i, \theta)\right\},$$

where $n = [r]$.

Proof. Let

$$\mathcal{M} = \max\left\{6, \sum_{i=1}^{6} \mathcal{N}_\theta(n+i, \theta), \sum_{i=1}^{6} \mathcal{N}_\theta(n-i, \theta)\right\}.$$

Then the two right-angled triangles with vertices

$$\begin{array}{lll} \mathbf{u}^{(1)}(r,\theta) & \text{and} & \mathbf{u}^{(1)}(r,\theta) + \mathcal{M}\mathbf{e}(\theta + \pi/2) \\ & \text{and} & \mathbf{u}^{(1)}(r,\theta) + \mathcal{M}\mathbf{e}(\theta + \pi/2) \pm 6\mathbf{e}(\theta), \end{array}$$

where $\mathbf{e}(\phi) = (\cos\phi, \sin\phi)$ for $\phi \in \mathbb{R}$, each contains a square of the type $A(m, n) \subseteq V_1$, in view of the convexity of V. The result follows from the definitions of $I(r, \theta)$ and $I^{(1)}(r, \theta)$ (see (65)–(67) and (71)). ♣

Proof of Lemma 12. Note that for every $n = -6, \ldots, -1, 0, 1, \ldots, [M(\theta)]$, we have

$$\mathcal{N}_\theta(n) \le \left|\mathbf{u}^{(1)}(n, \theta) - \mathbf{u}^{(1)}(n+1, \theta)\right|.$$

It follows from Lemma 13 that

$$\begin{aligned} \int_0^{M^*(\theta)} \operatorname{card}\left(I^{(1)}(r, \theta)\right) \mathrm{d}r\mathrm{d}\theta &\ll M^*(\theta) + \sum_{i=-6}^{6} \left|\mathbf{u}^{(1)}(n+i, \theta) - \mathbf{u}^{(1)}(n+i+1, \theta)\right| \\ &\le M^*(\theta) + 13 \,\text{perimeter}\,(V) \ll N^{1/2}. \end{aligned}$$

A similar argument applies for $I^{(2)}(r, \theta)$. ♣

It now follows from (68), (69) and Lemma 11 that

$$\begin{aligned} &\int_0^{\pi/4} \int_0^{M^*(\theta)} |E[\mathcal{P}; S(r, \theta)]| \mathrm{d}r\mathrm{d}\theta \\ &\qquad \ll N^{1/2} + \int_0^{\pi/4} \int_0^{M^*(\theta)} |E[\mathcal{P}; S_1(r, \theta)]| \mathrm{d}r\mathrm{d}\theta. \qquad (74) \end{aligned}$$

Next, we investigate the integral

$$\int_0^{\pi/4} \int_{M^*(\theta)}^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta.$$

Suppose that $0 \leq \theta \leq \pi/4$ and $M^*(\theta) \leq r \leq M(\theta)$. We shall use Lemma 7. If $M^*(\theta) \geq M(\theta) - 48$, then clearly $M(\theta) - r \leq 48$. On the other hand, $l(r,\theta) \ll N^{1/2}$ trivially. We now use the simple estimate

$$|E[\mathcal{P}; S(r,\theta)]| \ll \mu(S(r,\theta)) \leq (M(\theta) - r) l(r,\theta).$$

Clearly

$$\int_{M^*(\theta)}^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \ll N^{1/2}.$$

Suppose now that $M^*(\theta) < M(\theta) - 48$. Note that

$$\begin{aligned} |E[\mathcal{P}; S(r,\theta)]| &\ll \mu\left(\bigcup_{\substack{m,n \in \mathbb{Z} \\ A(m,n) \subseteq V_1 \\ A(m,n) \cap S(r,\theta) \neq \emptyset \\ A(m,n) \setminus S(r,\theta) \neq \emptyset}} A(m,n) \right) + \mu\left(\bigcup_{\substack{j=1 \\ R_j \cap S(r,\theta) \neq \emptyset \\ R_j \setminus S(r,\theta) \neq \emptyset}}^{L} R_j \right) \\ &\ll l(r,\theta) \ll 1 \end{aligned}$$

in view of Lemmas 7 and 8. It now follows that

$$\int_{M^*(\theta)}^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \ll M(\theta) - M^*(\theta) \ll N^{1/2}.$$

In either case,

$$\int_0^{\pi/4} \int_{M^*(\theta)}^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll N^{1/2}. \tag{75}$$

Combining (74) and (75), we get

$$\begin{aligned} &\int_0^{\pi/4} \int_0^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \\ &\quad \ll N^{1/2} + \int_0^{\pi/4} \int_0^{M^*(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta. \end{aligned} \tag{76}$$

As before, combining Lemmas 1 and 10 and (70), we have

$$\int_{-1/2}^{1/2} \int_{-1/2}^{1/2} \int_0^{\pi/4} \int_1^{M^*(\theta)} |E[\mathcal{P}; S_1(\mathbf{y}; r,\theta)]| \mathrm{d}r \mathrm{d}\theta \mathrm{d}y_1 \mathrm{d}y_2 \ll N^{1/2} (\log N)^2. \tag{77}$$

The estimate

$$\int_0^{\pi/4} \int_0^{M^*(\theta)} |E[\mathcal{P}; S_1(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll N^{1/2} (\log N)^2 \tag{78}$$

now follows from (77) in the same way that (61) follows from (58). Combining (76) and (78), we have

$$\int_0^{\pi/4} \int_0^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll N^{1/2} (\log N)^2.$$

Similarly, for $j = 1, \ldots, 7$, we have

$$\int_{j\pi/4}^{(j+1)\pi/4} \int_0^{M(\theta)} |E[\mathcal{P}; S(r,\theta)]| \mathrm{d}r \mathrm{d}\theta \ll N^{1/2} (\log N)^2.$$

Inequality (62) now follows.

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