Irregularities of point distribution relative to convex polygons II

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1. Introduction

Suppose that $\mathcal{P}$ is a distribution of N points in the unit square $U=[0,1]^2$, treated as a torus. For every measurable set B in U, let $Z[\mathcal{P};B]$ denote the number of points of $\mathcal{P}$ in B, and write

$$D[\mathcal{P};B]=Z[\mathcal{P};B]-N\mu(B),$$

where μ denotes the usual measure in $\mathbb{R}^2$.

Let $A\subseteq U$ be a closed convex polygon of diameter not exceeding 1 and centred at the origin $\mathbf{0}$. For every real number r satisfying $0\le r\le 1$ and for every angle θ satisfying $0\le\theta\le 2\pi$, let $\mathbf{v}=\theta(\mathbf{u})$ denote

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}=\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix}, \tag{1}$$

where $\mathbf{v}=(v_1,v_2)$ and $\mathbf{u}=(u_1,u_2)$, and write

$$A(r,\theta)=\{r\mathbf{v}:\mathbf{v}=\theta(\mathbf{u})\text{ for some }\mathbf{u}\in A\}; \tag{2}$$

in other words, $A(r,\theta)$ is obtained from A by first rotating clockwise by angle θ and then contracting by factor r about the origin $\mathbf{0}$. For every $\mathbf{x}\in U$, let

$$A(\mathbf{x},r,\theta)=\{\mathbf{x}+\mathbf{v}:\mathbf{v}\in A(r,\theta)\}, \tag{3}$$

so that $A(\mathbf{x},r,\theta)$ is a similar copy of A, with centre of gravity at $\mathbf{x}$.

In 1987, Beck [**1**] proved that

$$\inf_{|\mathcal{P}|=N}\sup_{\substack{\mathbf{x}\in U\\ 0\le r\le 1\\ 0\le\theta\le 2\pi}}|D[\mathcal{P};A(\mathbf{x},r,\theta)]|\gg_A N^{1/4},$$

where the supremum is taken over all similar copies $A(\mathbf{x},r,\theta)$ of A, and the infimum is taken over all distributions $\mathcal{P}$ of N points in U. In fact, Beck proved

Theorem A. *For every distribution $\mathcal{P}$ of N points in U, we have*

$$\int_0^1\int_0^{2\pi}\int_U |D[\mathcal{P};A(\mathbf{x},r,\theta)]|^2\mathrm{d}\mathbf{x}\mathrm{d}\theta\mathrm{d}r\gg_A N^{1/2}.$$

This is complemented by the result below, which can be proved using probabilistic methods (see, for example, Beck and Chen [**2**]).

Theorem B. *For every natural number N, there exists a distribution $\mathcal{P}$ of N points in U such that*

$$\int_0^1 \int_0^{2\pi} \int_U |D[\mathcal{P}; A(\mathbf{x}, r, \theta)]|^2 \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll_A N^{1/2}.$$

The purpose of this paper is to study the L^1–norm of the discrepancy function $D[\mathcal{P}; A(\mathbf{x}, r, \theta)]$. In particular, we prove

Theorem. *For every natural number $N \geq 2$, there exists a distribution $\mathcal{P}$ of N points in U such that*

$$\int_0^1 \int_0^{2\pi} \int_U |D[\mathcal{P}; A(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll_A (\log N)^2.$$

Our work in this paper is motivated by our study of irregularities of point distribution relative to half–planes in [**3**]. There, the same surprising difference between the L^1–norm and the L^2–norm of the corresponding discrepancy function is also present. In fact, the analogy between the two problems becomes clear on noting that a convex polygon is the intersection of a finite number of half–planes. We can therefore employ some of the techniques in [**3**].

In §2, we shall study the problem when N is a perfect square. Here the argument is relatively straightforward. However, we shall introduce some extra technicalities here for use in §3, where we extend the argument to when N is no longer a perfect square.

For ease of notation, we consider the following renormalized version of the problem. Let V be the square $[0, N^{1/2}]^2$, again treated as a torus (modulo $N^{1/2}$ for each coordinate). For every finite distribution $\mathcal{P}$ of points in V and every measurable set B in V, let $Z[\mathcal{P}; B]$ denote the number of points of $\mathcal{P}$ in B, and write

$$E[\mathcal{P}; B] = Z[\mathcal{P}; B] - \mu(B).$$

Let $A \subseteq V$ be a closed convex polygon of diameter not exceeding $N^{1/2}$ and centred at the origin $\mathbf{0}$. For every real number r satisfying $0 \leq r \leq 1$, every angle θ satisfying $0 \leq \theta \leq 2\pi$ and every $\mathbf{x} \in V$, we define $A(\mathbf{x}, r, \theta)$ in terms of (1)–(3). We shall prove

Main Theorem. *For every natural number $N \geq 2$, there exists a distribution $\mathcal{P}$ of N points in V such that*

$$\int_0^1 \int_0^{2\pi} \int_V |E[\mathcal{P}; A(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll_A N(\log N)^2. \tag{4}$$

The key idea in the proof of our Main Theorem is to split the integral over V in (4) into a sum of integrals over sets whose diameters are very small. We may then use the variable $\mathbf{x}$ in the same way as the probabilistic variable in Roth's probabilistic method in [**4**].

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2. A special case

We first of all consider the case when $N = M^2$, where $M \in \mathbb{N}$. We shall show that the set

$$\mathcal{P} = \{(m - 1/2, n - 1/2) : m, n \in \mathbb{N} \text{ and } 1 \leq m, n \leq M\}$$

of N points in V satisfies the inequality (4).

Let $A \subseteq V$ be a closed convex polygon of k sides and of diameter not exceeding M. Suppose that $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are the vertices of A, where

$$(\mathbf{v}_j - \mathbf{v}_{j-1}) \cdot \mathbf{e}(\theta_j + \pi/2) = |\mathbf{v}_j - \mathbf{v}_{j-1}|,$$

with $0 \leq \theta_1 < \ldots < \theta_k < 2\pi$ and $\mathbf{v}_0 = \mathbf{v}_k$. Here $\mathbf{e}(\theta) = (\cos\theta, \sin\theta)$ and $\mathbf{u} \cdot \mathbf{v}$ denotes the scalar product of $\mathbf{u}$ and $\mathbf{v}$. Let T_j denote the side of A with vertices $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$, and note that the perpendicular from $\mathbf{0}$ to T_j makes an angle θ_j with the positive x_1–axis.

Consider now the set $A(\mathbf{x}, r, \theta)$, where the contraction $r \in [0, 1]$, the rotation $\theta \in [0, 2\pi]$ and the centre of gravity $\mathbf{x} \in V$ are fixed. We let $\mathbf{v}_1(\mathbf{x}, r, \theta), \ldots, \mathbf{v}_k(\mathbf{x}, r, \theta)$ denote respectively the images of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ after contraction r, rotation θ and translation $\mathbf{x}$. For each $j = 1, \ldots, k$, let $T_j(\mathbf{x}, r, \theta)$ denote the line segment joining the vertices $\mathbf{v}_{j-1}(\mathbf{x}, r, \theta)$ and $\mathbf{v}_j(\mathbf{x}, r, \theta)$. For convenience, we use the convention that $T_j(\mathbf{x}, r, \theta)$ does not include either of these two vertices. Furthermore, let

$$T_j^*(\mathbf{x}, r, \theta) = \bigcup_{\substack{i=1 \\ i \neq j}}^{k} T_j(\mathbf{x}, r, \theta);$$

in other words, $T_j^*(\mathbf{x}, r, \theta)$ is the union of all the other edges of $A(\mathbf{x}, r, \theta)$. Now, for every $m, n \in \mathbb{N}$ satisfying $1 \leq m, n \leq M$, let

$$B(m, n) = (m - 1, m] \times (n - 1, n].$$

We shall write, for each $j = 1, \ldots, k$,

$$S_j(\mathbf{x}, r, \theta) = \bigcup_{\substack{1 \leq m, n \leq M \\ B(m,n) \cap T_j(\mathbf{x}, r, \theta) \neq \emptyset \\ B(m,n) \cap T_j^*(\mathbf{x}, r, \theta) = \emptyset}} B(m, n);$$

in other words, $S_j(\mathbf{x}, r, \theta)$ is the union of all squares $B(m, n)$ that intersect with the edge $T_j(\mathbf{x}, r, \theta)$. Furthermore, let

$$V(\mathbf{x}, r, \theta) = \left(\bigcup_{\substack{1 \le m,n \le M \\ B(m,n) \cap A(\mathbf{x},r,\theta) \neq \emptyset \\ B(m,n) \not\subseteq A(\mathbf{x},r,\theta)}} B(m, n) \right) \setminus \left(\bigcup_{j=1}^{k} S_j(\mathbf{x}, r, \theta) \right);$$

in other words, $V(\mathbf{x}, r, \theta)$ is the union of all $B(m, n)$ that intersect non–trivially with more than one edge of $A(\mathbf{x}, r, \theta)$. We also let

$$W_0(\mathbf{x}, r, \theta) = \bigcup_{\substack{1 \le m,n \le M \\ B(m,n) \cap A(\mathbf{x},r,\theta) = \emptyset}} B(m, n)$$

and

$$W_1(\mathbf{x}, r, \theta) = \left(\bigcup_{\substack{1 \le m,n \le M \\ B(m,n) \subseteq A(\mathbf{x},r,\theta)}} B(m, n) \right) \setminus \left(\bigcup_{j=1}^{k} S_j(\mathbf{x}, r, \theta) \right).$$

Note that

$$(0, M]^2 = \left(\bigcup_{j=1}^{k} S_j(\mathbf{x}, r, \theta) \right) \cup V(\mathbf{x}, r, \theta) \cup \left(\bigcup_{i=0}^{1} W_i(\mathbf{x}, r, \theta) \right),$$

so that, for every $j = 1, \ldots, k$, writing

$$A_j(\mathbf{x}, r, \theta) = A(\mathbf{x}, r, \theta) \cap S_j(\mathbf{x}, r, \theta),$$

we have

$$\begin{aligned} A(\mathbf{x}, r, \theta) = & \left(\bigcup_{j=1}^{k} A_j(\mathbf{x}, r, \theta) \right) \cup (A(\mathbf{x}, r, \theta) \cap V(\mathbf{x}, r, \theta)) \\ & \cup \left(\bigcup_{i=0}^{1} (A(\mathbf{x}, r, \theta) \cap W_i(\mathbf{x}, r, \theta)) \right). \end{aligned} \tag{5}$$

Note now that the $(k + 3)$ sets on the right–hand side of (5) are pairwise disjoint, so that

$$\begin{aligned} E[\mathcal{P}; A(\mathbf{x}, r, \theta)] = & \sum_{j=1}^{k} E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)] + E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap V(\mathbf{x}, r, \theta)] \\ & + \sum_{i=0}^{1} E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap W_i(\mathbf{x}, r, \theta)]. \end{aligned} \tag{6}$$

It is easy to see that for $i = 0, 1$, if $B(m,n) \subseteq W_i(\mathbf{x}, r, \theta)$, then $Z[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap B(m,n)] = \mu(A(\mathbf{x}, r, \theta) \cap B(m,n)) = i$, so that $E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap B(m,n)] = 0$. It follows that

$$E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap W_i(\mathbf{x}, r, \theta)] = 0 \tag{7}$$

for $i = 0, 1$. On the other hand, note that

$$\text{card}\left(\left\{(m,n) \in \mathbb{N}^2 : B(m,n) \subseteq V(\mathbf{x}, r, \theta)\right\}\right) \ll_A 1,$$

so that

$$E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap V(\mathbf{x}, r, \theta)] \ll_A 1. \tag{8}$$

Combining (6)–(8), it is easy to see that to prove (4), it suffices to prove that for every $j = 1, \ldots, k$, we have

$$\int_0^1 \int_0^{2\pi} \int_V |E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll N(\log N)^2. \tag{9}$$

Note that we may extend the definition of $A_j(\mathbf{x}, r, \theta)$ by periodicity 2π on θ. Suppose now that $0 \leq \theta + \theta_j \leq \pi/4$. Let

$$I_j(\mathbf{x}, r, \theta) = \{n \in \mathbb{N} : B(m,n) \subseteq S_j(\mathbf{x}, r, \theta) \text{ for some } m \in \mathbb{N}\}.$$

Consider the edge $T_j(\mathbf{x}, r, \theta)$. Since V is treated as a torus, $T_j(\mathbf{x}, r, \theta) \cap [0, M]^2$ can be interpreted as the union of at most three line segments of the form

$$T_{j,\rho}(\mathbf{x}, r, \theta) \qquad (\rho = 1, 2, 3)$$

in the square $[0, M]^2$, with equation

$$(\mathbf{y} + N^{1/2}(\alpha_\rho, \beta_\rho) - \mathbf{x}) \cdot \mathbf{e}(\theta + \theta_j) = rM_j, \tag{10}$$

where $\mathbf{y} = (y_1, y_2)$ denotes any arbitrary point on $T_{j,\rho}(\mathbf{x}, r, \theta)$, where M_j denotes the perpendicular distance of T_j from $\mathbf{0}$, and where $\alpha_\rho, \beta_\rho \in \{-1, 0, 1\}$. Here we use $N^{1/2}$ instead of M, as we need the greater generality in §3.

Let $PT_{j,\rho}(\mathbf{x}, r, \theta)$ denote the projection of $T_{j,\rho}(\mathbf{x}, r, \theta)$ on the y_2–axis, and let

$$I_{j,\rho}(\mathbf{x}, r, \theta) = \{n \in (0, M] : (n-1, n] \subseteq PT_{j,\rho}(\mathbf{x}, r, \theta)\} \cap I_j(\mathbf{x}, r, \theta).$$

Clearly

$$I_{j,\rho}(\mathbf{x}, r, \theta) \subseteq I_j(\mathbf{x}, r, \theta) \qquad (\rho = 1, 2, 3)$$

and

$$\text{card}\left(I_j(\mathbf{x}, r, \theta) \setminus \left(\bigcup_{\rho=1}^{3} I_{j,\rho}(\mathbf{x}, r, \theta)\right)\right) \leq 1. \tag{11}$$

For each $\rho = 1, 2, 3$, we now write

$$S_{j,\rho}(\mathbf{x}, r, \theta) = \bigcup_{\substack{1 \le m \le M \\ n \in I_{j,\rho}(\mathbf{x},r,\theta) \\ B(m,n) \subseteq S_j(\mathbf{x},r,\theta)}} B(m, n).$$

Then

$$\bigcup_{\rho=1}^{3} S_{j,\rho}(\mathbf{x}, r, \theta) \subseteq S_j(\mathbf{x}, r, \theta),$$

so that writing

$$A_{j,\rho}(\mathbf{x}, r, \theta) = A(\mathbf{x}, r, \theta) \cap S_{j,\rho}(\mathbf{x}, r, \theta),$$

we have

$$\bigcup_{\rho=1}^{3} A_{j,\rho}(\mathbf{x}, r, \theta) \subseteq A_j(\mathbf{x}, r, \theta). \tag{12}$$

Note now that the union on the left–hand side of (12) is pairwise disjoint. Furthermore, in view of (11), the set

$$A_j(\mathbf{x}, r, \theta) \setminus \bigcup_{\rho=1}^{3} A_{j,\rho}(\mathbf{x}, r, \theta)$$

is contained in the union of at most two squares of the form $B(m, n)$, so that

$$\left| E\left[\mathcal{P}; A_j(\mathbf{x}, r, \theta) \setminus \bigcup_{\rho=1}^{3} A_{j,\rho}(\mathbf{x}, r, \theta) \right] \right| \le 2.$$

It follows that

$$|E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)]| \le \sum_{\rho=1}^{3} |E[\mathcal{P}; A_{j,\rho}(\mathbf{x}, r, \theta)]| + 2. \tag{13}$$

Let $\mathbf{n} = (n_1, n_2) \in \mathbb{N}^2 \cap (0, M]^2$, and let $\rho = 1, 2, 3$. We consider the set $A_{j,\rho}(\mathbf{n}, r, \theta)$. For each $n \in I_{j,\rho}(\mathbf{n}, r, \theta)$, let

$$A_{j,\rho}(n; \mathbf{n}, r, \theta) = A_{j,\rho}(\mathbf{n}, r, \theta) \cap (\mathbb{R} \times (n - 1, n]).$$

There exists a smallest integer m such that $B(m, n) \cap A_{j,\rho}(n; \mathbf{n}, r, \theta) \neq \emptyset$. Then it follows from (10) that

$$\begin{aligned} &Z[\mathcal{P}; A_{j,\rho}(n; \mathbf{n}, r, \theta)] \\ &\quad = [rM_j \sec\phi - (n + N^{1/2}\beta_\rho - n_2 - 1/2)\tan\phi - N^{1/2}\alpha_\rho + n_1 - m + 3/2], \end{aligned}$$

where $\phi = \theta + \theta_j$. On the other hand,

$$\begin{aligned} &\mu(A_{j,\rho}(n; \mathbf{n}, r, \theta)]) \\ &\quad = rM_j \sec\phi - (n + N^{1/2}\beta_\rho - n_2 - 1/2)\tan\phi - N^{1/2}\alpha_\rho + n_1 - m + 1. \end{aligned}$$

Clearly

$$\begin{aligned}&E[\mathcal{P}; A_{j,\rho}(n; \mathbf{n}, r, \theta)]\\&\qquad = -\psi(rM_j \sec\phi - (n + N^{1/2}\beta_\rho - n_2 - 1/2)\tan\phi - N^{1/2}\alpha_\rho + n_1 + 1/2),\end{aligned}$$

where $\psi(z) = z - [z] - 1/2$ for every $z \in \mathbb{R}$. It follows that

$$\begin{aligned}&E[\mathcal{P}; A_{j,\rho}(\mathbf{n}, r, \theta)]\\&= -\sum_{n\in I_{j,\rho}(\mathbf{n},r,\theta)} \psi(rM_j \sec\phi - (n + N^{1/2}\beta_\rho - n_2 - 1/2)\tan\phi - N^{1/2}\alpha_\rho + n_1 + 1/2).\end{aligned}$$

Suppose now that $\mathbf{x} \in B(\mathbf{n})$. Then $\mathbf{x} = \mathbf{n} - \mathbf{z}$, where $\mathbf{z} = (z_1, z_2) \in [0,1)^2$. By permuting $I_{j,\rho}(\mathbf{x}, r, \theta)$ for $\rho = 1, 2, 3$ if necessary, we may assume that

$$\mathrm{card}(I_{j,\rho}(\mathbf{x}, r, \theta) \triangle I_{j,\rho}(\mathbf{n}, r, \theta)) \ll 1$$

for $\rho = 1, 2, 3$. Let

$$G[\mathcal{P}; j, \rho; \mathbf{x}, r, \theta] = -\sum_{n\in I_{j,\rho}(\mathbf{n},r,\theta)} \psi(\mathcal{H}(r, M_j, \phi, N, \alpha_\rho, \beta_\rho, \mathbf{n}, z_2) - z_1 - n\tan\phi),$$

where

$$\begin{aligned}&\mathcal{H}(r, M_j, \phi, N, \alpha_\rho, \beta_\rho, \mathbf{n}, z_2)\\&\qquad = rM_j \sec\phi - (N^{1/2}\beta_\rho - n_2 + z_2 - 1/2)\tan\phi - N^{1/2}\alpha_\rho + n_1 + 1/2.\end{aligned}$$

Then

$$E[\mathcal{P}; A_{j,\rho}(\mathbf{x}, r, \theta)] - G[\mathcal{P}; j, \rho; \mathbf{x}, r, \theta] \ll 1. \tag{14}$$

The function $\psi(z) = z - [z] - 1/2$ has Fourier expansion

$$-\sum_{\nu\neq 0} \frac{e(z\nu)}{2\pi \mathrm{i}\nu}.$$

It follows that the Fourier expansion of $G[\mathcal{P}; j, \rho; \mathbf{x}, r, \theta]$ is given by

$$\sum_{\nu\neq 0} \frac{e(\nu\mathcal{H}(r, M_j, \phi, N, \alpha_\rho, \beta_\rho, \mathbf{n}, z_2))}{2\pi \mathrm{i}\nu} \sum_{n\in I_{j,\rho}(\mathbf{n},r,\theta)} e(-n\nu\tan\phi)e(-z_1\nu).$$

Hence for every $x_2 \in (n_2 - 1, n_2]$, we have, by Parseval's theorem, that

$$\int_{n_1-1}^{n_1} |G[\mathcal{P}; j, \rho; \mathbf{x}, r, \theta]|^2 \mathrm{d}x_1 \ll \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n\in I_{j,\rho}(\mathbf{n},r,\theta)} e(n\nu\tan\phi) \right|^2,$$

so that, on noting that $I_{j,\rho}(\mathbf{n},r,\theta)$ is a set of consecutive integers, we have

$$\int_{B(\mathbf{n})} |G[\mathcal{P};j,\rho;\mathbf{x},r,\theta]|^2 \mathrm{d}\mathbf{x} \ll \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \min\{M^2, \|\nu\tan\phi\|^2\}, \tag{15}$$

where $\|\beta\| = \min_{n\in\mathbb{Z}} |\beta - n|$ for every $\beta \in \mathbb{R}$.

We need the following crucial estimate.

Lemma 1. *We have*

$$\int_0^{\pi/4} \left(\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \min\{M^2, \|\nu\tan\phi\|^2\}\right)^{1/2} \mathrm{d}\phi \ll (\log M)^2.$$

Proof. Since $\tan\phi \asymp \phi$ if $0 \le \phi \le \pi/4$, it suffices to show that

$$\int_0^1 \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \min\{M^2, \|n\omega\|^{-2}\}\right)^{1/2} \mathrm{d}\omega \ll (\log M)^2. \tag{16}$$

Clearly

$$\sum_{n=1}^{\infty} \frac{1}{n^2} \min\{M^2, \|n\omega\|^{-2}\} \le \sum_{n=1}^{M^2} \frac{1}{n^2} \min\{M^2, \|n\omega\|^{-2}\} + 1,$$

so that

$$\left(\sum_{n=1}^{\infty} \frac{1}{n^2} \min\{M^2, \|n\omega\|^{-2}\}\right)^{1/2} \le \sum_{n=1}^{M^2} \frac{1}{n} \min\{M, \|n\omega\|^{-1}\} + 1. \tag{17}$$

Now, for every $n = 1, \ldots, M^2$, we have

$$\int_0^1 \min\{M, \|n\omega\|^{-1}\}\mathrm{d}\omega = 2n \int_0^{1/2n} \min\{M, (n\omega)^{-1}\}\mathrm{d}\omega \ll \log M. \tag{18}$$

Inequality (16) now follows on combining (17) and (18). ♣

By the Cauchy–Schwarz inequality, we have

$$\int_{B(\mathbf{n})} |G[\mathcal{P};j,\rho;\mathbf{x},r,\theta]| \mathrm{d}\mathbf{x} \ll \left(\int_{B(\mathbf{n})} |G[\mathcal{P};j,\rho;\mathbf{x},r,\theta]|^2 \mathrm{d}\mathbf{x}\right)^{1/2}. \tag{19}$$

It follows from (13)–(15), (19) and Lemma 1 that

$$\int_0^1 \int_0^{\pi/4} \int_{B(\mathbf{n})} |E[\mathcal{P};A_j(\mathbf{x},r,\theta)]| \mathrm{d}\mathbf{x}\mathrm{d}\phi\mathrm{d}r \ll (\log N)^2.$$

Similarly, for $i = 1, \ldots, 7$, we have

$$\int_0^1 \int_{i\pi/4}^{(i+1)\pi/4} \int_{B(\mathbf{n})} |E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\phi \mathrm{d}r \ll (\log N)^2.$$

It follows that

$$\int_0^1 \int_0^{2\pi} \int_{B(\mathbf{n})} |E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll (\log N)^2.$$

Summing over all $B(\mathbf{n})$ in V, we have

$$\begin{aligned}
&\int_0^1 \int_0^{2\pi} \int_{[0,M]^2} |E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \\
&= \sum_{\mathbf{n} \in \mathbb{N}^2 \cap (0,M]^2} \int_0^1 \int_0^{2\pi} \int_{B(\mathbf{n})} |E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \\
&\ll N(\log N)^2.
\end{aligned}$$

Inequality (9) follows.

3. Proof of the Main Theorem

Suppose now that the natural number $N \geq 2$. Let $M \in \mathbb{N}$ be chosen such that $M^2 \leq N < (M+1)^2$. Consider the square $V = [0, N^{1/2}]^2$. Clearly $[0, M]^2 \subseteq V$. Let $Q = N^{1/2} - M$. For every $i = 1, \ldots, R$, where $R = [MQ]$, let

$$C_1(i) = \left[\frac{i-1}{Q}, \frac{i}{Q}\right] \times [M, N^{1/2}] \qquad \text{and} \qquad C_2(i) = [M, N^{1/2}] \times \left[\frac{i-1}{Q}, \frac{i}{Q}\right],$$

and let

$$C_1 = \bigcup_{i=1}^{R} C_1(i) \qquad \text{and} \qquad C_2 = \bigcup_{i=1}^{R} C_2(i).$$

Furthermore, let

$$C_3 = V \setminus \left([0, M]^2 \cup C_1 \cup C_2\right),$$

and note that

$$\mu(C_3) \in \{0, 1, 2\}, \tag{20}$$

and that $\mu(C_3) = 0$ if and only if $N = M^2$.

Let $A \subseteq V$ be a closed convex polygon of k sides, of diameter not exceeding $N^{1/2}$ and with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

We now let

$$\mathcal{P}_1 = \{(m - 1/2, n - 1/2) : m, n \in \mathbb{N} \text{ and } 1 \le m, n \le M\}$$

as in the special case. Furthermore, let

$$\begin{aligned}\mathcal{P}_2 = &\left\{ \left(\frac{i}{Q} - \frac{1}{2Q}, M + \frac{Q}{2} \right) : i = 1, \ldots, R \right\} \\ &\cup \left\{ \left(M + \frac{Q}{2}, \frac{i}{Q} - \frac{1}{2Q} \right) : i = 1, \ldots, R \right\}\end{aligned}$$

and let $\mathcal{P}_3$ be any set of $\mu(C_3)$ points in C_3. We shall show that the set $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ satisfies (4).

Consider now the set $A(\mathbf{x}, r, \theta)$, where the contraction $r \in [0, 1]$, the rotation $\theta \in [0, 2\pi]$ and the centre of gravity $\mathbf{x} \in V$ are fixed. As before, we let $\mathbf{v}_1(\mathbf{x}, r, \theta), \ldots, \mathbf{v}_k(\mathbf{x}, r, \theta)$ denote respectively the images of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ after contraction r, rotation θ and translation $\mathbf{x}$. For each $j = 1, \ldots, k$, we define $T_j(\mathbf{x}, r, \theta)$, $S_j(\mathbf{x}, r, \theta)$, $V(\mathbf{x}, r, \theta)$, $W_0(\mathbf{x}, r, \theta)$, $W_1(\mathbf{x}, r, \theta)$ and $A_j(\mathbf{x}, r, \theta)$ as before. Then

$$\begin{aligned}A(\mathbf{x}, r, \theta) = &\left(\bigcup_{j=1}^{k} A_j(\mathbf{x}, r, \theta) \right) \cup (A(\mathbf{x}, r, \theta) \cap V(\mathbf{x}, r, \theta)) \\ &\cup \left(\bigcup_{i=0}^{1} (A(\mathbf{x}, r, \theta) \cap W_i(\mathbf{x}, r, \theta)) \right) \\ &\cup \left(\bigcup_{i=1}^{3} (A(\mathbf{x}, r, \theta) \cap C_i) \right). \end{aligned} \tag{21}$$

Note now that the $(k+6)$ sets on the right–hand side of (21) satisfy $\mu(B_1 \cap B_2) = 0$ and $B_1 \cap B_2 \cap \mathcal{P} = \emptyset$. It follows that

$$\begin{aligned}E[\mathcal{P}; A(\mathbf{x}, r, \theta)] = &\sum_{j=1}^{k} E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)] + E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap V(\mathbf{x}, r, \theta)] \\ &+ \sum_{i=0}^{1} E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap W_i(\mathbf{x}, r, \theta)] \\ &+ \sum_{i=1}^{3} E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap C_i]. \end{aligned} \tag{22}$$

Suppose first of all that $\mathbf{x} \in [0, M]^2$. Then as in §2, writing

$$\begin{aligned}F[\mathcal{P}; A(\mathbf{x}, r, \theta)] = &\sum_{j=1}^{k} E[\mathcal{P}; A_j(\mathbf{x}, r, \theta)] + E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap V(\mathbf{x}, r, \theta)] \\ &+ \sum_{i=0}^{1} E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap W_i(\mathbf{x}, r, \theta)],\end{aligned}$$

we can show that

$$\int_0^1 \int_0^{2\pi} \int_{[0,M]^2} |F[\mathcal{P}; A(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll_A N(\log N)^2.$$

To prove (4), it therefore remains to show that for every $i = 1, 2, 3$, we have

$$\int_0^1 \int_0^{2\pi} \int_{[0,M]^2} |E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap C_i]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll_A N; \tag{23}$$

and that

$$\int_0^1 \int_0^{2\pi} \int_{V \setminus [0,M]^2} |E[\mathcal{P}; A(\mathbf{x}, r, \theta)]| \mathrm{d}\mathbf{x} \mathrm{d}\theta \mathrm{d}r \ll_A N. \tag{24}$$

In view of (20), we must have $|E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap C_3]| \leq 2$, so that (23) holds when $i = 3$. Consider now $E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap C_1]$. Let r and θ be fixed, and let $\mathcal{E} = \{0, R/Q\} \times [M, N^{1/2}]$; in other words, $\mathcal{E}$ denotes the two short edges of the rectangle C_1. The following three lemmas are obvious. For every $\mathbf{x} \in [0, M]^2$, let $\partial A(\mathbf{x}, r, \theta)$ denotes the boundary of $A(\mathbf{x}, r, \theta)$.

Lemma 2. *Let $\mathbf{x} \in [0, M]^2$. Suppose that*

$$\partial A(\mathbf{x}, r, \theta) \cap \mathcal{E} = \emptyset. \tag{25}$$

Suppose further that no vertex of $A(\mathbf{x}, r, \theta)$ lies in C_1; in other words,

$$\mathbf{v}_j(\mathbf{x}, r, \theta) \notin C_1 \qquad (j = 1, \ldots, k). \tag{26}$$

Then $|E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap C_1]| \leq 1$.

Lemma 3. *We have $\mu(\{\mathbf{x} \in [0, M]^2 : \partial A(\mathbf{x}, r, \theta) \cap \mathcal{E} \neq \emptyset\}) \leq kN^{1/2}$.*

Lemma 4. *For every $j = 1, \ldots, k$, we have $\mu(\{\mathbf{x} \in [0, M]^2 : \mathbf{v}_j(\mathbf{x}, r, \theta) \in C_1\}) \leq \mu(C_1) \leq N^{1/2}$.*

Note also that

$$|E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap C_1]| \leq \mu(C_1) \leq N^{1/2} \tag{27}$$

trivially. Let $\chi(r, \theta) = \{\mathbf{x} \in [0, M]^2 : \text{(25) and (26) hold}\}$. Then by Lemmas 2–4 and (27), we have

$$\int_{[0,M]^2} |E[\mathcal{P}; A(\mathbf{x}, r, \theta) \cap C_1]| \mathrm{d}\mathbf{x} \leq \int_{\chi(r,\theta)} \mathrm{d}\mathbf{x} + \int_{[0,M]^2 \setminus \chi(r,\theta)} N^{1/2} \mathrm{d}\mathbf{x} \leq (2k+1)N.$$

Inequality (23) follows for $i = 1$. A similar argument applies in the case $i = 2$.

Suppose now that $\mathbf{x} \in V \setminus [0,M]^2$. Clearly $E[\mathcal{P};B] \ll N^{1/2}$ for every set B in the union (21). On the other hand, $\mu(V \setminus [0,M]^2) \ll N^{1/2}$. Inequality (24) clearly follows on noting (22).

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