Irregularities of point distribution relative to convex polygons III

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To Wolfgang Schmidt on the occasion of his 60th birthday

1. Introduction

Suppose that $\mathcal{P}$ is a distribution of N points in the unit square $U = [0,1]^2$. For every $\mathbf{x} = (x_1, x_2) \in U$, let $B(\mathbf{x}) = [0, x_1] \times [0, x_2]$ denote the aligned rectangle containing all points $\mathbf{y} = (y_1, y_2) \in U$ satisfying $0 \leq y_1 \leq x_1$ and $0 \leq y_2 \leq x_2$. Denote by $Z[\mathcal{P}; B(\mathbf{x})]$ the number of points of $\mathcal{P}$ that lie in $B(\mathbf{x})$, and consider the discrepancy function

$$D[\mathcal{P}; B(\mathbf{x})] = Z[\mathcal{P}; B(\mathbf{x})] - N\mu(B(\mathbf{x})),$$

where μ denotes the usual area measure.

The following two results are classical.

Theorem A1. (Roth [**8**]) *There exists an absolute constant $c_1 > 0$ such that for every natural number N and every distribution $\mathcal{P}$ of N points in U, we have*

$$\int_U |D[\mathcal{P}; B(\mathbf{x})]|^2 \mathrm{d}\mathbf{x} > c_1 \log N.$$

Theorem A2. (Davenport [**6**]) *There exists an absolute constant $c_2 > 0$ such that for every natural number $N \geq 2$, there exists a distribution $\mathcal{P}$ of N points in U such that*

$$\int_U |D[\mathcal{P}; B(\mathbf{x})]|^2 \mathrm{d}\mathbf{x} < c_2 \log N.$$

The situation is somewhat different when the aligned rectangles are replaced by similar copies of a given convex polygon. More precisely, suppose that $\mathcal{P}$ is a distribution of N points in the unit square $U = [0,1]^2$, treated as a torus. Suppose that

$A \subseteq U$ is a closed convex polygon of diameter less than 1 and with centre of gravity at the origin $\mathbf{0}$. For every real number r satisfying $0 \leq r \leq 1$ and every angle θ satisfying $0 \leq \theta \leq 2\pi$, let $\mathbf{v} = \theta(\mathbf{u})$ denote

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$, and write

$$A(r,\theta) = \{r\mathbf{v} : \mathbf{v} = \theta(\mathbf{u}) \text{ for some } \mathbf{u} \in A\};$$

in other words, $A(r,\theta)$ is obtained from A by first rotating clockwise by angle θ and then contracting by factor r about the origin $\mathbf{0}$. For every $\mathbf{x} \in U$, let

$$A(\mathbf{x}, r, \theta) = \{\mathbf{x} + \mathbf{v} : \mathbf{v} \in A(r,\theta)\},$$

so that $A(\mathbf{x}, r, \theta)$ is a similar copy of A, with centre of gravity at $\mathbf{x}$. Denote by $Z[\mathcal{P}; A(\mathbf{x}, r, \theta)]$ the number of points of $\mathcal{P}$ that lie in $A(\mathbf{x}, r, \theta)$, and consider the discrepancy function

$$D[\mathcal{P}; A(\mathbf{x}, r, \theta)] = Z[\mathcal{P}; A(\mathbf{x}, r, \theta)] - N\mu(A(\mathbf{x}, r, \theta)).$$

Corresponding to Theorems A1 and A2, we have the following two results.

Theorem B1. (Beck [**1**]) *There exists a constant $c_3 = c_3(A) > 0$ such that for every natural number N and every distribution $\mathcal{P}$ of N points in U, we have*

$$\int_0^{2\pi} \int_0^1 \int_U |D[\mathcal{P}; A(\mathbf{x}, r, \theta)]|^2 \mathrm{d}\mathbf{x}\mathrm{d}r\mathrm{d}\theta > c_3(A) N^{1/2}.$$

Theorem B2. (Beck and Chen [**3**]) *There exists a constant $c_4 = c_4(A) > 0$ such that for every natural number N, there exists a distribution $\mathcal{P}$ of N points in U such that*

$$\int_0^{2\pi} \int_0^1 \int_U |D[\mathcal{P}; A(\mathbf{x}, r, \theta)]|^2 \mathrm{d}\mathbf{x}\mathrm{d}r\mathrm{d}\theta < c_4(A) N^{1/2}.$$

Let us now study the case when rotation is not present. More precisely, suppose that $\mathcal{P}$ is a distribution of N points in the unit square $U = [0,1]^2$, treated as a torus. Suppose that $A \subseteq U$ is a closed convex polygon of diameter less than 1 and with centre of gravity at the origin $\mathbf{0}$. For every real number r satisfying $0 \leq r \leq 1$ and every $\mathbf{x} \in U$, let

$$A(\mathbf{x}, r) = \{\mathbf{x} + r\mathbf{u} : \mathbf{u} \in A\},$$

so that $A(\mathbf{x},r)$ is a homothetic copy of A, with centre of gravity at $\mathbf{x}$. Denote by $Z[\mathcal{P};A(\mathbf{x},r)]$ the number of points of $\mathcal{P}$ that lie in $A(\mathbf{x},r)$, and consider the discrepancy function

$$D[\mathcal{P};A(\mathbf{x},r)]=Z[\mathcal{P};A(\mathbf{x},r)]-N\mu(A(\mathbf{x},r)).$$

Corresponding to Theorems A1 and B1, we have the following lower bound result.

Theorem C1. (Beck [**2**]) *There exists a constant $c_5=c_5(A)>0$ such that for every natural number N and every distribution $\mathcal{P}$ of N points in U, we have*

$$\int_0^1\int_U |D[\mathcal{P};A(\mathbf{x},r)]|^2\mathrm{d}\mathbf{x}\mathrm{d}r>c_5(A)\log N.$$

The purpose of this paper is to prove that Theorem C1 is best possible, apart from the value of the constant $c_5=c_5(A)$. We prove the following analogue of Theorems A2 and B2.

Theorem C2. *There exists a constant $c_6=c_6(A)>0$ such that for every natural number $N\geq 2$, there exists a distribution $\mathcal{P}$ of N points in U such that*

$$\int_0^1\int_U |D[\mathcal{P};A(\mathbf{x},r)]|^2\mathrm{d}\mathbf{x}\mathrm{d}r<c_6(A)\log N.$$

Our work in this paper is motivated by our study of irregularities of point distribution relative to similar copies of a closed convex polygon A. In [**5**], we showed that there exists a constant $c_7=c_7(A)>0$ such that for every natural number $N\geq 2$, there exists a distribution $\mathcal{P}$ of N points in the unit torus U such that

$$\int_0^{2\pi}\int_0^1\int_U |D[\mathcal{P};A(\mathbf{x},r,\theta)]|\mathrm{d}\mathbf{x}\mathrm{d}r\mathrm{d}\theta<c_7(A)(\log N)^2.$$

In the proof, the point set $\mathcal{P}$ that we constructed is made up primarily of a square lattice, with appropriate adjustments near the edge of U. It is clear that the resulting discrepancy function $D[\mathcal{P};A(\mathbf{x},r,\theta)]$ can be rather large in magnitude for some values of θ and rather small in magnitude for other values of θ.

This observation leads us to consider, in our present case, the possibility of rotating a square lattice to a suitable angle, and then perhaps make appropriate adjustments near the edge of U in the same spirit as in our earlier work [**4**,**5**] on irregularities of point distribution relative to half planes and convex polygons. Rotating a square lattice to a suitable angle presents no difficulties, and we appeal to a result of Davenport [**7**] on diophantine approximation. However, the analysis of the adjusted point set near the edge of U appears to give rise to an error term too large for the method to succeed, at least in the case when one of the sides of A is parallel to a side of U.

Instead, we shall use Roth's probabilistic method first introduced in [**9**], introduce an extra translation variable and consider some average of the discrepancy function over a collection of translated copies of our basic construction. Note first that this approach will not give explicitly any point set $\mathcal{P}$ that will satisfy the conclusion of Theorem C2. On the other hand, if the collection of translated copies of the basic construction is too "small", then we cannot use Parseval's theorem and study the coefficients arising from the Fourier series of the discrepancy function. However, if the collection of translated copies of the basic construction is large enough to enable us to use Parseval's theorem in an appropriate way, then we may end up with a point set which does not contain the correct number of points. In §6, we shall discuss a technique to overcome this difficulty.

For the sake of simplicity of notation, we consider instead a distribution $\mathcal{P}$ of N points in the square $V = [0, N^{1/2}]^2$, treated as a torus. Suppose that $A \subseteq V$ is a closed convex polygon of diameter less than $N^{1/2}$ and with centre of gravity at the origin $\mathbf{0}$. For every real number r satisfying $0 \le r \le 1$ and every $\mathbf{x} \in V$, let

$$A(\mathbf{x}, r) = \{\mathbf{x} + r\mathbf{u} : \mathbf{u} \in A\},$$

so that $A(\mathbf{x}, r)$ is a homothetic copy of A, with centre of gravity at $\mathbf{x}$. Denote by $Z[\mathcal{P}; A(\mathbf{x}, r)]$ the number of points of $\mathcal{P}$ that lie in $A(\mathbf{x}, r)$, and consider the discrepancy function

$$E[\mathcal{P}; A(\mathbf{x}, r)] = Z[\mathcal{P}; A(\mathbf{x}, r)] - \mu(A(\mathbf{x}, r)).$$

Theorem C2 follows immediately from the result below.

Main Theorem. *There exists a constant $c_8 = c_8(A) > 0$ such that for every natural number $N \ge 2$, there exists a distribution $\mathcal{P}$ of N points in V such that*

$$\int_0^1 \int_V |E[\mathcal{P}; A(\mathbf{x}, r)]|^2 \mathrm{d}\mathbf{x}\mathrm{d}r < c_8(A) N \log N.$$

2. The Point Set

Let $A \subseteq V$ be a closed convex polygon of diameter less than $N^{1/2}$ and with centre of gravity at the origin $\mathbf{0}$. Throughout this paper, A is fixed, and constants that arise from all subsequent discussion may depend on this choice of A.

Suppose that the polygon A has k sides, with vertices $\mathbf{v}_1, \ldots, \mathbf{v}_k$, where

$$(\mathbf{v}_j - \mathbf{v}_{j-1}) \cdot \mathbf{e}(\theta_j + \pi/2) = |\mathbf{v}_j - \mathbf{v}_{j-1}|,$$

with $0 \le \theta_1 < \ldots < \theta_k < 2\pi$ and $\mathbf{v}_0 = \mathbf{v}_k$. Here $\mathbf{e}(\theta) = (\cos\theta, \sin\theta)$ and $\mathbf{u} \cdot \mathbf{v}$ denotes the scalar product of $\mathbf{u}$ and $\mathbf{v}$. Let T_j denote the side of A with vertices $\mathbf{v}_{j-1}$ and $\mathbf{v}_j$, and note that the perpendicular from $\mathbf{0}$ to T_j makes an angle θ_j with the positive x_1–axis.

Recall that a real number β is said to be badly approximable if there exists a constant $c_9 = c_9(\beta) > 0$ such that $\nu\|\nu\beta\| > c_9(\beta)$ for every positive integer ν. Here $\|x\| = \min_{n\in\mathbb{Z}} |x - n|$ denotes the distance of x from the nearest integer.

To construct the point set $\mathcal{P}$ which will satisfy the conclusion of the Main Theorem, we first of all need the following result on diophantine approximation.

Lemma 1. (Davenport [**7**]) *Suppose that $f_1, \ldots, f_r$ are real–valued functions of a real variable, and have continuous first derivatives in some open interval I containing α_0, where $f_1'(\alpha_0), \ldots, f_r'(\alpha_0)$ are all non–zero. Then there exists $\alpha \in I$ such that $f_1(\alpha), \ldots, f_r(\alpha)$ are all badly approximable.*

An immediate conseqeunce of Lemma 1 is the following.

Lemma 2. *There exists a real number $\alpha \in [0, 2\pi)$ such that the $k + 2$ numbers*

$$\tan\alpha, \tan(\alpha + \pi/2), \tan(\alpha + \theta_1), \ldots, \tan(\alpha + \theta_k)$$

are all finite and badly approximable.

We now choose a value α that satisfies the conclusion of Lemma 2 and keep it fixed.

Consider the square lattice

$$\Lambda = \mathbb{Z}^2 = \{(n_1, n_2) : n_1, n_2 \in \mathbb{Z}\}$$

with determinant 1. We shall rotate Λ clockwise by angle α to obtain the lattice

$$\Lambda_\alpha = \{\mathbf{v} : \mathbf{v} = \alpha(\mathbf{u}) \text{ for some } \mathbf{u} \in \Lambda\}.$$

Here $\mathbf{v} = \alpha(\mathbf{u})$ denotes

$$\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix},$$

where $\mathbf{v} = (v_1, v_2)$ and $\mathbf{u} = (u_1, u_2)$. For every $\mathbf{w} \in \mathbb{R}^2$, write

$$\mathbf{w} + \Lambda_\alpha = \{\mathbf{w} + \mathbf{v} : \mathbf{v} \in \Lambda_\alpha\}.$$

In other words, the lattice $\mathbf{w} + \Lambda_\alpha$ is obtained from the lattice Λ by first rotating clockwise by angle α and then translating by $\mathbf{w}$. Note that $\mathbf{w} + \Lambda_\alpha$ is a square lattice with determinant 1.

Our first goal is show that there is a suitable $\mathbf{w} \in \mathbb{R}^2$ such that the set

$$\mathcal{P} = (\mathbf{w} + \Lambda_\alpha) \cap V$$

satisfies a weaker form of the Main Theorem. However, our technique in §§4–5 cannot guarantee that this set contains exactly N points.

It is rather inconvenient to work with the lattice Λ_α. We shall therefore rotate V, A as well as Λ_α anticlockwise by angle α about the origin $\mathbf{0}$ to obtain V', A' and Λ respectively.

For every $\mathbf{w} \in \mathbb{R}^2$, let

$$\mathbf{w} + \Lambda = \{\mathbf{w} + \mathbf{v} : \mathbf{v} \in \Lambda\},$$

and write

$$\mathcal{P}_\mathbf{w} = (\mathbf{w} + \Lambda) \cap V'.$$

Note that while $\mathcal{P}_\mathbf{w}$ contains "on average" N points, it is possible that the number of points of $\mathcal{P}_\mathbf{w}$ can differ from N for some $\mathbf{w} \in \mathbb{R}^2$.

3. An Averaging Argument

Denote by $\mathrm{POL}(\alpha; \theta_1, \ldots, \theta_k; N)$ the collection of all convex polygons $B \subseteq V'$ whose edges are parallel to the edges of A' and V', so that the angle each edge of B makes with the positive x_1–axis belongs to the collection

$$\mathcal{A} = \{\alpha, \alpha + \pi/2, \alpha + \theta_1, \ldots, \alpha + \theta_k\}.$$

In this section and §4, it is convenient not to treat V' as a torus.

Lemma 3. *There exists a constant $c_{10} = c_{10}(\alpha; \theta_1, \ldots, \theta_k) > 0$ such that for every convex polygon $B \in \mathrm{POL}(\alpha; \theta_1, \ldots, \theta_k; N)$, we have*

$$\int_{[0,1]^2} |E[\mathcal{P}_\mathbf{w}; B]|^2 \mathrm{d}\mathbf{w} \leq c_{10}(\alpha; \theta_1, \ldots, \theta_k) \log N.$$

For every $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$ and every $\mathbf{w} = (w_1, w_2) \in [0,1]^2$, let

$$S(\mathbf{n}, \mathbf{w}) = (n_1 + w_1 - 1/2, n_1 + w_1 + 1/2] \times (n_2 + w_2 - 1/2, n_2 + w_2 + 1/2];$$

in other words, $S(\mathbf{n}, \mathbf{w})$ is a square of area 1, centred at $\mathbf{n} + \mathbf{w}$ and with sides parallel to the coordinate axes.

Suppose now that $B \in \mathrm{POL}(\alpha; \theta_1, \ldots, \theta_k; N)$ is chosen and fixed. Let $T_1, \ldots, T_m$ be the edges of B, and, for every $j = 1, \ldots, m$, let

$$T_j^* = \bigcup_{\substack{i=1 \\ i \neq j}}^{m} T_i;$$

in other words, T_j^* denotes the union of all the edges of B except the edge T_j.

Consider the set

$$\mathcal{N} = \{\mathbf{n} \in \mathbb{Z}^2 : S(\mathbf{n},\mathbf{w}) \cap B \neq \emptyset \text{ for some } \mathbf{w} \in [0,1]^2\}.$$

Clearly for every $\mathbf{w} \in [0,1]^2$, we have

$$E[\mathcal{P}_{\mathbf{w}}; B] = \sum_{\mathbf{n} \in \mathcal{N}} E[\mathcal{P}_{\mathbf{w}}; S(\mathbf{n},\mathbf{w}) \cap B]. \tag{1}$$

Next, note that $\mathcal{N}$ can be expressed as a pairwise disjoint union

$$\mathcal{N} = \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_m \cup \mathcal{N}^+ \cup \mathcal{N}^-, \tag{2}$$

where, for every $j = 1, \ldots, m$, we have

$$\begin{aligned}\mathcal{N}_j = \{\mathbf{n} \in \mathcal{N} : S(\mathbf{n},\mathbf{w}) \cap T_j \neq \emptyset \text{ for some } \mathbf{w} \in [0,1]^2 \\ \text{and } S(\mathbf{n},\mathbf{w}) \cap T_j^* = \emptyset \text{ for every } \mathbf{w} \in [0,1]^2\},\end{aligned}$$

and where

$$\begin{aligned}\mathcal{N}^+ = \{\mathbf{n} \in \mathcal{N} : \text{ there exist } j', j'' \in \{1, \ldots, m\} \text{ with } j' \neq j'' \\ \text{and } \mathbf{w}', \mathbf{w}'' \in [0,1]^2 \text{ such that} \\ S(\mathbf{n},\mathbf{w}') \cap T_{j'} \neq \emptyset \text{ and } S(\mathbf{n},\mathbf{w}'') \cap T_{j''} \neq \emptyset\}\end{aligned}$$

and

$$\mathcal{N}^- = \{\mathbf{n} \in \mathcal{N} : S(\mathbf{n},\mathbf{w}) \cap T_j = \emptyset \text{ for every } \mathbf{w} \in [0,1]^2 \text{ and } j = 1, \ldots, m\}.$$

For every $\mathbf{n} \in \mathcal{N}^-$ and every $\mathbf{w} \in [0,1]^2$, we clearly have $S(\mathbf{n},\mathbf{w}) \subseteq B$, so that

$$Z[\mathcal{P}_{\mathbf{w}}; S(\mathbf{n},\mathbf{w}) \cap B] = \mu(S(\mathbf{n},\mathbf{w}) \cap B) = 1.$$

It follows that

$$\sum_{\mathbf{n} \in \mathcal{N}^-} E[\mathcal{P}_{\mathbf{w}}; S(\mathbf{n},\mathbf{w}) \cap B] = 0. \tag{3}$$

On the other hand,

$$E[\mathcal{P}_{\mathbf{w}}; S(\mathbf{n},\mathbf{w}) \cap B] = O(1)$$

always. Since $|\mathcal{N}^+| = O_A(1)$, it follows that

$$\sum_{\mathbf{n} \in \mathcal{N}^+} E[\mathcal{P}_{\mathbf{w}}; S(\mathbf{n},\mathbf{w}) \cap B] = O_A(1). \tag{4}$$

Combining (1)–(4), we have

$$E[\mathcal{P}_{\mathbf{w}};B]=\sum_{j=1}^{m}\sum_{\mathbf{n}\in\mathcal{N}_j}E[\mathcal{P}_{\mathbf{w}};S(\mathbf{n},\mathbf{w})\cap B]+O_A(1).$$

For every $j=1,\ldots,m$, write

$$E_j[\mathcal{P}_{\mathbf{w}};B]=\sum_{\mathbf{n}\in\mathcal{N}_j}E[\mathcal{P}_{\mathbf{w}};S(\mathbf{n},\mathbf{w})\cap B].$$

Since $m=O_A(1)$, Lemma 3 will follow if we show that for every $j=1,\ldots,m$, we have

$$\int_{[0,1]^2}|E_j[\mathcal{P}_{\mathbf{w}};B]|^2\mathrm{d}\mathbf{w}=O_A(\log N). \tag{5}$$

4. A Fourier Series Technique

Suppose that the edge T_j of the polygon B lies on the line

$$(y_1-z_1,y_2-z_2)\cdot(\cos\phi,\sin\phi)=0, \tag{6}$$

where $\phi\in\mathcal{A}$, z_1,z_2 are constants and $\mathbf{y}=(y_1,y_2)$ denotes an arbitrary point on the line. In view of symmetry, we may assume, without loss of generality, that $0\le\phi\le\pi/4$. Then (6) can be written in the form

$$y_1=z_1-(y_2-z_2)\tan\phi.$$

Write

$$I_j=\{n_2\in\mathbb{Z}:(n_1,n_2)\in\mathcal{N}_j\text{ for some }n_1\in\mathbb{Z}\}.$$

Suppose that $n\in I_j$. Let $h\in\mathbb{Z}$ be smallest such that $(h,n)\in\mathcal{N}_j$. Then it is not too difficult to see that

$$\sum_{\substack{\mathbf{n}\in\mathcal{N}_j\\ n_2=n}}Z[\mathcal{P}_{\mathbf{w}};S(\mathbf{n},\mathbf{w})\cap B]=[z_1-(n+w_2-z_2)\tan\phi-h-w_1+1]$$

and

$$\sum_{\substack{\mathbf{n}\in\mathcal{N}_j\\ n_2=n}}\mu(S(\mathbf{n},\mathbf{w})\cap B)=z_1-(n+w_2-z_2)\tan\phi-h-w_1+\frac{1}{2},$$

so that

$$\sum_{\substack{\mathbf{n}\in\mathcal{N}_j\\ n_2=n}}E[\mathcal{P}_{\mathbf{w}};S(\mathbf{n},\mathbf{w})\cap B]=-\psi(z_1-(n+w_2-z_2)\tan\phi-w_1),$$

where $\psi(u) = u - [u] - 1/2$ for every $u \in \mathbb{R}$. It follows that

$$E_j[\mathcal{P}_{\mathbf{w}}; B] = -\sum_{n \in I_j} \psi(z_1 - (n + w_2 - z_2)\tan\phi - w_1).$$

The function $\psi(u) = u - [u] - 1/2$ has the Fourier expansion

$$-\sum_{\nu \neq 0} \frac{e(\nu u)}{2\pi \mathrm{i}\nu},$$

where $e(x) = \mathrm{e}^{2\pi \mathrm{i} x}$ for every $x \in \mathbb{R}$. It follows that the Fourier expansion of $E_j[\mathcal{P}_{\mathbf{w}}; B]$ is given by

$$\sum_{\nu \neq 0} \frac{e(\nu(z_1 + z_2 \tan\phi))}{2\pi \mathrm{i}\nu} \sum_{n \in I_j} e(-\nu(n + w_2)\tan\phi) e(-\nu w_1).$$

For every $w_2 \in [0,1]$, we have, by Parseval's theorem, that

$$\begin{aligned}\int_0^1 |E_j[\mathcal{P}_{\mathbf{w}}; B]|^2 \mathrm{d}w_1 &\ll \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n \in I_j} e(-\nu(n + w_2)\tan\phi) \right|^2 \\ &= \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n \in I_j} e(-\nu n \tan\phi) \right|^2,\end{aligned}$$

so that on integrating with respect to w_2, we obtain

$$\int_{[0,1]^2} |E_j[\mathcal{P}_{\mathbf{w}}; B]|^2 \mathrm{d}\mathbf{w} \ll \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n \in I_j} e(-\nu n \tan\phi) \right|^2. \tag{7}$$

We next note that I_j is a collection of consecutive integers, and use the hypothesis that $\tan\phi$ is badly approximable. The estimate below is due to Davenport [**6**]. For the sake of completeness, we include the short proof here.

Lemma 4. *We have*

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n \in I_j} e(-\nu n \tan\phi) \right|^2 \ll_\phi \log(2|I_j|). \tag{8}$$

Proof. It is well known that

$$\left| \sum_{n \in I_j} e(-\nu n \tan\phi) \right| \ll \min\{|I_j|, \|\nu \tan\phi\|^{-1}\},$$

where $\|x\| = \min_{n\in\mathbb{Z}} |x-n|$ denotes the distance of x from the nearest integer. It follows that

$$S = \sum_{\nu=1}^{\infty} \frac{1}{\nu^2} \left| \sum_{n\in I_j} e(-\nu n \tan\phi) \right|^2 \ll \sum_{h=1}^{\infty} 2^{-2h} \sum_{2^{h-1}\leq \nu < 2^h} \min\{|I_j|^2, \|\nu \tan\phi\|^{-2}\}.$$

Since $\tan\phi$ is badly approximable, there exists a constant $c_{11} = c_{11}(\phi) > 0$ such that

$$\nu \|\nu \tan\phi\| > c_{11}(\phi) \tag{9}$$

for every positive integer ν. Note that this implies that if $2^{h-1} \leq \nu < 2^h$, then

$$\|\nu \tan\phi\| > c_{11}(\phi) 2^{-h}.$$

On the other hand, for any pair $h, p \in \mathbb{N}$, there are at most two values of ν satisfying $2^{h-1} \leq \nu < 2^h$ and

$$p c_{11}(\phi) 2^{-h} \leq \|\nu \tan\phi\| < (p+1) c_{11}(\phi) 2^{-h};$$

for otherwise the difference $(\nu_1 - \nu_2)$ of two of them would contradict (9). It follows that

$$\begin{aligned}
S &\ll_\phi \sum_{h=1}^{\infty} \sum_{p=1}^{\infty} \min\{2^{-2h}|I_j|^2, p^{-2}\} \\
&= \sum_{2^h \leq |I_j|} \sum_{p=1}^{\infty} \min\{2^{-2h}|I_j|^2, p^{-2}\} + \sum_{2^h > |I_j|} \sum_{p=1}^{\infty} \min\{2^{-2h}|I_j|^2, p^{-2}\} \\
&\ll \sum_{2^h \leq |I_j|} \sum_{p=1}^{\infty} p^{-2} + \sum_{2^h > |I_j|} \left(2^{-2h}|I_j|^2 2^h |I_j|^{-1} + \sum_{p > 2^h |I_j|^{-1}} p^{-2} \right) \\
&\ll \sum_{2^h \leq |I_j|} 1 + \sum_{2^h > |I_j|} 2^{-h}|I_j| \ll \log(2|I_j|)
\end{aligned}$$

as required. ♣

Note now that for every $j = 1, \ldots, m$, we have $|I_j| = O(N^{1/2})$. The inequality (5) follows immediately on combining (7) and (8). Lemma 3 now follows.

5. A Weaker Version of the Main Theorem

For every $\mathbf{x} \in V'$ and every real number r satisfying $0 \leq r \leq 1$, the convex polygon $A'(\mathbf{x}, r)$ is the union of at most four polygons in $\mathrm{POL}(\alpha; \theta_1, \ldots, \theta_k; N)$. It follows from Lemma 3 that

$$\int_{[0,1]^2} |E[\mathcal{P}_{\mathbf{w}}; A'(\mathbf{x}, r)]|^2 \mathrm{d}\mathbf{w} \leq 4 c_{10}(\alpha; \theta_1, \ldots, \theta_k) \log N. \tag{10}$$

Clearly

$$\int_0^1 \int_{V'} \int_{[0,1]^2} |E[\mathcal{P}_{\mathbf{w}}; A'(\mathbf{x}, r)]|^2 \mathrm{d}\mathbf{w}\mathrm{d}\mathbf{x}\mathrm{d}r \leq 4c_{10}(\alpha; \theta_1, \ldots, \theta_k) N \log N.$$

We can therefore conclude that there exists $\mathbf{w}_0 \in [0,1]^2$ such that the set $\mathcal{P}_{\mathbf{w}_0}$ of points in V' satisfies

$$\int_0^1 \int_{V'} |E[\mathcal{P}_{\mathbf{w}_0}; A'(\mathbf{x}, r)]|^2 \mathrm{d}\mathbf{x}\mathrm{d}r \leq 4c_{10}(\alpha; \theta_1, \ldots, \theta_k) N \log N.$$

Unfortunately, this gives only a weaker version of the Main Theorem, since we cannot guarantee that the set $\mathcal{P}_{\mathbf{w}_0}$ has exactly N points.

6. Completion of the Proof

For every $\mathbf{w} \in [0,1]^2$, let $\mathcal{P}^*_{\mathbf{w}}$ be obtained from $\mathcal{P}_{\mathbf{w}}$ in the following way: If $\mathcal{P}_{\mathbf{w}}$ has exactly N points, then take $\mathcal{P}^*_{\mathbf{w}} = \mathcal{P}_{\mathbf{w}}$. If $\mathcal{P}_{\mathbf{w}}$ has more than N points, then we remove a suitable number of points from $\mathcal{P}_{\mathbf{w}}$ to obtain a set $\mathcal{P}^*_{\mathbf{w}}$ with exactly N points. If $\mathcal{P}_{\mathbf{w}}$ has fewer than N points, then we add a suitable number of points to $\mathcal{P}_{\mathbf{w}}$ to obtain a set $\mathcal{P}^*_{\mathbf{w}}$ with exactly N points.

Suppose now that for every $\mathbf{w} \in [0,1]^2$, a set $\mathcal{P}^*_{\mathbf{w}}$ has been determined and contains exactly N points. Then clearly

$$Z[\mathcal{P}^*_{\mathbf{w}}; V'] = N = \mu(V'),$$

so that

$$Z[\mathcal{P}_{\mathbf{w}}; V'] - Z[\mathcal{P}^*_{\mathbf{w}}; V'] = E[\mathcal{P}_{\mathbf{w}}; V']. \tag{11}$$

On the other hand, for every $\mathbf{x} \in V'$ and every real number r satisfying $0 \leq r \leq 1$, we clearly have

$$|Z[\mathcal{P}_{\mathbf{w}}; A'(\mathbf{x}, r)] - Z[\mathcal{P}^*_{\mathbf{w}}; A'(\mathbf{x}, r)]| \leq |Z[\mathcal{P}_{\mathbf{w}}; V'] - Z[\mathcal{P}^*_{\mathbf{w}}; V']|. \tag{12}$$

It follows from (11) and (12) that

$$\begin{aligned}
|E[\mathcal{P}^*_{\mathbf{w}}; A'(\mathbf{x}, r)]| &= |Z[\mathcal{P}^*_{\mathbf{w}}; A'(\mathbf{x}, r)] - \mu(A'(\mathbf{x}, r))| \\
&\leq |Z[\mathcal{P}_{\mathbf{w}}; A'(\mathbf{x}, r)] - \mu(A'(\mathbf{x}, r))| + |Z[\mathcal{P}^*_{\mathbf{w}}; A'(\mathbf{x}, r)] - Z[\mathcal{P}_{\mathbf{w}}; A'(\mathbf{x}, r)]| \\
&\leq |E[\mathcal{P}_{\mathbf{w}}; A'(\mathbf{x}, r)]| + |E[\mathcal{P}_{\mathbf{w}}; V']|.
\end{aligned} \tag{13}$$

Clearly $V' \in \mathrm{POL}(\alpha; \theta_1, \ldots, \theta_k; N)$. It follows from Lemma 3 that

$$\int_{[0,1]^2} |E[\mathcal{P}_{\mathbf{w}}; V']|^2 \mathrm{d}\mathbf{w} \leq c_{10}(\alpha; \theta_1, \ldots, \theta_k) \log N. \tag{14}$$

Combining (10), (13) and (14), we conclude that

$$\int_{[0,1]^2} |E[\mathcal{P}^*_{\mathbf{w}}; A'(\mathbf{x},r)]|^2 \mathrm{d}\mathbf{w} \le 5c_{10}(\alpha;\theta_1,\ldots,\theta_k)\log N.$$

Clearly

$$\int_0^1 \int_{V'} \int_{[0,1]^2} |E[\mathcal{P}^*_{\mathbf{w}}; A'(\mathbf{x},r)]|^2 \mathrm{d}\mathbf{w}\mathrm{d}\mathbf{x}\mathrm{d}r \le 5c_{10}(\alpha;\theta_1,\ldots,\theta_k)N\log N.$$

We can now conclude that there exists $\mathbf{w}_1 \in [0,1]^2$ such that the set $\mathcal{P}^*_{\mathbf{w}_1}$ contains exactly N points in V' and satisfies

$$\int_0^1 \int_{V'} |E[\mathcal{P}^*_{\mathbf{w}_1}; A'(\mathbf{x},r)]|^2 \mathrm{d}\mathbf{x}\mathrm{d}r \le 5c_{10}(\alpha;\theta_1,\ldots,\theta_k)N\log N.$$

The Main Theorem now follows easily.

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