DISCREPANCY WITH RESPECT TO CONVEX POLYGONS

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To Henryk Woźniakowski on the occasion of his 60th birthday

ABSTRACT. We study the problem of discrepancy of finite point sets in the unit square with respect to convex polygons, when the directions of the edges are fixed, when the number of edges is bounded, as well as when no such restrictions are imposed. In all three cases, we obtain estimates for the supremum norm that are very close to best possible.

1. INTRODUCTION

Suppose that $\mathcal{P}$ is a distribution of $N > 1$ points, not necessarily distinct, in the unit square $[0,1]^2$. For every Lebesgue measurable set $A \subseteq [0,1]^2$, let $Z[\mathcal{P}; A]$ denote the number of points of $\mathcal{P}$ that fall into A, and consider the discrepancy function

$$D[\mathcal{P}; A] = Z[\mathcal{P}; A] - N\mu(A), \tag{1}$$

where $\mu(A)$ denotes the measure (or area) of A. We shall study the discrepancy function (1) when the subsets A are closed convex polygons in $[0,1]^2$. More precisely, we study the behaviour of the function

$$\sup_{A \in \mathcal{A}} |D[\mathcal{P}; A]|$$

with respect to three classes $\mathcal{A}$ of convex polygons in $[0,1]^2$.

Notation. We adopt standard Vinogradov notation. For two functions f and g, we write $f \ll g$ to denote the existence of a positive constant c such that $|f| \leq cg$. For any non-negative functions f and g, we write $f \gg g$ to denote the existence of a positive constant c such that $f \geq cg$. The inequality signs $\ll$ and $\gg$ may be used with subscripts involving parameters such as k and Θ, in which case the positive constant c in question may depend on the parameters indicated.

Let $\Theta = (\theta_1, \ldots, \theta_k)$, where $\theta_1, \ldots, \theta_k \in [0, \pi)$ are fixed. We denote by $\mathcal{A}(\Theta)$ the collection of all convex polygons A in $[0,1]^2$ such that every side of A makes an angle θ_i for some $i = 1, \ldots, k$ with the positive horizontal axis. Note that if $\Theta = (0, \pi/2)$, then $\mathcal{A}(\Theta)$ is simply the collection of all aligned rectangles in $[0,1]^2$. Then the famous result of Schmidt [12] shows that for every set $\mathcal{P}$ of N points in $[0,1]^2$, we have

$$\sup_{A \in \mathcal{A}(0,\pi/2)} |D[\mathcal{P}; A]| \gg \log N. \tag{2}$$

This result is best possible, apart from the implicit constant in the inequality, as an old result of Lerch [10] implies that there exists a set $\mathcal{P}$ of N points in $[0,1]^2$ such that

$$\sup_{A \in \mathcal{A}(0,\pi/2)} |D[\mathcal{P}; A]| \ll \log N.$$

For the general case, the ideas in Beck and Chen [4] can be adapted easily to show that for every set $\mathcal{P}$ of N points in $[0,1]^2$, we have

$$\sup_{A \in \mathcal{A}(\Theta)} |D[\mathcal{P}; A]| \gg_\Theta \log N.$$

Here we establish the following complementary result.

Theorem 1. *Suppose that* $\Theta = (\theta_1, \ldots, \theta_k)$, *where* $\theta_1, \ldots, \theta_k \in [0, \pi)$ *are fixed. Then for every integer* $N > 1$, *there exists a set* $\mathcal{P}$ *of* N *points in* $[0, 1]^2$ *such that*

$$\sup_{A \in \mathcal{A}(\Theta)} |D[\mathcal{P}; A]| \ll_\Theta \log N.$$

Next, we relax the restriction on the direction of the sides of the convex polygons and replace this with a restriction on the number of sides instead. We denote by $\mathcal{A}_k$ the collection of all convex polygons in $[0, 1]^2$ with at most k sides. Then a result of Beck [1] implies that for every set $\mathcal{P}$ of N points in $[0, 1]^2$, we have

$$\sup_{A \in \mathcal{A}_k} |D[\mathcal{P}; A]| \gg_k N^{1/4}. \tag{3}$$

Here we establish the following upper bound.

Theorem 2. *For every integer* $N > 1$, *there exists a set* $\mathcal{P}$ *of* N *points in* $[0, 1]^2$ *such that*

$$\sup_{A \in \mathcal{A}_k} |D[\mathcal{P}; A]| \ll_k N^{1/4} (\log N)^{1/2}. \tag{4}$$

Finally, we relax all the restrictions on the direction and number of sides of the convex polygons. Accordingly, we denote by $\mathcal{A}^*$ the collection of all convex polygons in $[0, 1]^2$. Our study is motivated by the wonderfully elegant work of Schmidt [13] and Beck [2] on the collection $\mathcal{C}^*$ of all convex sets in $[0, 1]^2$. Here, for every set $\mathcal{P}$ of N points in $[0, 1]^2$, we have

$$\sup_{A \in \mathcal{C}^*} |D[\mathcal{P}; A]| \gg N^{1/3}. \tag{5}$$

This is essentially best possible. For every integer $N > 1$, there exists a set $\mathcal{P}$ of N points in $[0, 1]^2$ such that

$$\sup_{A \in \mathcal{C}^*} |D[\mathcal{P}; A]| \ll N^{1/3} (\log N)^4.$$

Here we establish the following lower bound.

Theorem 3. *For every integer* $N > 1$, *for every set* $\mathcal{P}$ *of* N *points in* $[0, 1]^2$, *we have*

$$\sup_{A \in \mathcal{A}^*} |D[\mathcal{P}; A]| \gg N^{1/3}. \tag{6}$$

We remark that some of the arguments can be extended to polytopes in the d-dimensional unit cube $[0, 1]^d$. In particular, the inequalities (3) and (4) can be generalized to arbitrary dimensions d, with the exponent $1/4$ replaced by the exponent $1/2 - 1/2d$, while the inequalities (5) and (6) can also be generalized to arbitrary dimensions d, with the exponent $1/3$ replaced by the exponent $1 - 2/(d+1)$. On the other hand, the generalization of the inequality (2) to arbitrary dimensions is one of the most frustrating unsolved problems in the subject. For example, we do not know whether for every set $\mathcal{P}$ of N points in the cube $[0, 1]^3$, there is an aligned rectangular box A in $[0, 1]^3$ such that $|D[\mathcal{P}; A]| \gg (\log N)^2$.

2. Diophantine Approximation

To establish Theorem 1, we shall follow the argument of Beck and Chen [5] and make use of a suitably scaled and rotated copy of the lattice $\mathbb{Z}^2$. The rotation is made possible by the following result on diophantine approximation due to Davenport [7].

Lemma 2.1. *Suppose that $f_1, \ldots, f_r$ are real valued functions of a real variable, with continuous first derivatives in some open interval I containing some point $\alpha_0 \in \mathbb{R}$ such that $f_1'(\alpha_0), \ldots, f_r'(\alpha_0)$ are all non-zero. Then there exists $\alpha \in I$ such that $f_1(\alpha), \ldots, f_r(\alpha)$ are all badly approximable.*

Remark. A real number θ, such as $\theta = \sqrt{2}$, is said to be badly approximable if there exists a constant $c > 0$ such that $n\|n\theta\| > c$ for every natural number $n \in \mathbb{N}$. Here $\|\beta\|$ denotes the distance of β from the nearest integer.

More precisely, we shall use the following simple consequence.

Lemma 2.2. *Suppose that the angles $\theta_1, \ldots, \theta_k \in [0, \pi)$ are fixed. Then there exists $\alpha \in [0, 2\pi)$ such that*

$$\tan \alpha, \tan(\alpha - \pi/2), \tan(\alpha - \theta_1), \ldots, \tan(\alpha - \theta_k)$$

are all finite and badly approximable.

We shall be concerned with the collection $\mathcal{A}(\Theta)$ of convex polygons in $[0, 1]^2$, where $\theta_1, \ldots, \theta_k \in [0, \pi)$ are fixed. Recall that every side of such a polygon $A \in \mathcal{A}(\Theta)$ makes an angle θ_i for some $i = 1, \ldots, k$ with the positive horizontal axis.

Corresponding to the given Θ, we now choose a value of α from Lemma 2.2 and keep it fixed throughout. We would like to consider the lattice Λ formed by rotating the lattice $(N^{-1/2}\mathbb{Z})^2$ anticlockwise by the angle α about the origin. In particular, we are interested in the lattice points of Λ that fall into $[0, 1]^2$. Notationally, however, it is far simpler to rescale and rotate the unit square $[0, 1]^2$ and the convex polygons in $\mathcal{A}(\Theta)$. Accordingly, we consider the following rescaled and rotated variant of the original problem.

Let U denote the image of the square $[0, N^{1/2}]^2$ rotated clockwise by the angle α about the origin, and let $\mathcal{A}_N(\Theta; \alpha)$ denote the collection of all convex polygons B in U such that every side of B either is parallel to a side of U or makes an angle $\theta_i - \alpha$ for some $i = 1, \ldots, k$ with the positive horizontal axis. For every measurable subset $B \subseteq U$, let $Z(B)$ denote the number of lattice points of $\mathbb{Z}^2$ that fall into B, and write $E(B) = Z(B) - \mu(B)$. We need the following intermediate result.

Lemma 2.3. *For every $B \in \mathcal{A}_N(\Theta; \alpha)$, we have*

$$|E(B)| \ll_\Theta \log N.$$

Deduction of Theorem 1. Unfortunately, the set $\mathbb{Z}^2 \cap U$ does not necessarily have precisely N points. Let $\mathcal{Q}$ denote a set of precisely N points in U obtained by adding to or removing from $\mathbb{Z}^2 \cap U$ precisely $||\mathbb{Z}^2 \cap U| - N|$ points. Note that

$$||\mathbb{Z}^2 \cap U| - N| = |E(U)| \ll_\Theta \log N,$$

in view of Lemma 2.3. For every $B \in \mathcal{A}_N(\Theta; \alpha)$, we now let $Z[\mathcal{Q}; B]$ denote the number of points of $\mathcal{Q}$ in B. Then

$$
\begin{aligned}
|Z[\mathcal{Q}; B] - \mu(B)| &\leq |E(B)| + |Z(B) - Z[\mathcal{Q}; B]| \\
&\leq |E(B)| + |Z(U) - Z[\mathcal{Q}; U]| \\
&= |E(B)| + |E(U)| \\
&\ll_\Theta \log N.
\end{aligned}
$$

Now let $\mathcal{P}$ be obtained by rotating $N^{-1/2}\mathcal{Q}$ anticlockwise by the angle α. Then $\mathcal{P}$ is a set of precisely N points in $[0, 1]^2$, and the inequality

$$|D[\mathcal{P}; A]| \ll_\Theta \log N$$

holds for every convex polygon $A \in \mathcal{A}(\Theta)$. $\qquad\square$

Proof of Lemma 2.3. We adopt the convention that $\theta_1, \ldots, \theta_k$ are distinct, but note that no convex polygon can have three parallel sides. For every $\mathbf{n} = (n_1, n_2) \in \mathbb{Z}^2$, let

$$S(\mathbf{n}) = (n_1 - \tfrac{1}{2}, n_1 + \tfrac{1}{2}] \times (n_2 - \tfrac{1}{2}, n_2 + \tfrac{1}{2}].$$

For any convex polygon $B \in \mathcal{A}_N(\Theta; \alpha)$, let

$$\mathcal{N} = \{\mathbf{n} \in \mathbb{Z}^2 : S(\mathbf{n}) \cap B \neq \emptyset\},$$

so that

$$E(B) = \sum_{\mathbf{n} \in \mathcal{N}} E(B \cap S(\mathbf{n})).$$

Furthermore, for every $i = 1, \ldots, k$, let T_i denote the edge(s) of B that makes the angle $\theta_i - \alpha$ with the positive horizontal axis, let T_i^* denote the totality of all the other edges of B, and write

$$\mathcal{N}_i = \{\mathbf{n} \in \mathcal{N} : S(\mathbf{n}) \cap T_i \neq \emptyset \text{ and } S(\mathbf{n}) \cap T_i^* = \emptyset\}.$$

We also write

$$\mathcal{N}^+ = \{\mathbf{n} \in \mathcal{N} : \text{there exist } i' \neq i'' \text{ with } S(\mathbf{n}) \cap T_{i'} \neq \emptyset \text{ and } S(\mathbf{n}) \cap T_{i''} \neq \emptyset\}$$

and

$$\mathcal{N}^- = \{\mathbf{n} \in \mathcal{N} : S(\mathbf{n}) \cap T_i = \emptyset \text{ for every } i\}.$$

Clearly $\mathcal{N} = \mathcal{N}_1 \cup \ldots \cup \mathcal{N}_k \cup \mathcal{N}^+ \cup \mathcal{N}^-$, and

$$E(B) = \sum_{i=1}^{k} \sum_{\mathbf{n} \in \mathcal{N}_i} E(B \cap S(\mathbf{n})) + \sum_{\mathbf{n} \in \mathcal{N}^+} E(B \cap S(\mathbf{n})) + \sum_{\mathbf{n} \in \mathcal{N}^-} E(B \cap S(\mathbf{n})). \quad (7)$$

It is easy to see that $|\mathcal{N}^+| = O_\Theta(1)$ and that $|E(B \cap S(\mathbf{n}))| \leq 1$ for every $\mathbf{n} \in \mathcal{N}$, so that

$$\sum_{\mathbf{n} \in \mathcal{N}^+} E(B \cap S(\mathbf{n})) = O_\Theta(1). \quad (8)$$

It is also easy to see that

$$\sum_{\mathbf{n} \in \mathcal{N}^-} E(B \cap S(\mathbf{n})) = 0. \quad (9)$$

Combining (7)–(9), we conclude that

$$E(B) = \sum_{i=1}^{k} \sum_{\mathbf{n} \in \mathcal{N}_i} E(B \cap S(\mathbf{n})) + O_\Theta(1).$$

To prove Lemma 2.3, it remains to prove that for every $i = 1, \ldots, k$, we have

$$\sum_{\mathbf{n} \in \mathcal{N}_i} E(B \cap S(\mathbf{n})) \ll_\Theta \log N. \quad (10)$$

Write $\varphi_i = \theta_i - \alpha$. In view of symmetry, we may assume that $0 \leq \varphi_i \leq \pi/4$. There are at most two edges of B that makes the angle φ_i with the positive horizontal axis. Let one of these lie on the line

$$\frac{x_2 - a_2}{x_1 - a_1} = \tan \varphi_i,$$

where $(x_1, x_2) \in \mathbb{R}^2$ denotes any point on the line and where a_1 and a_2 are real constants. Elementary calculation then shows that the contribution from this edge to the sum in (10) is given by

$$\pm \sum_{A_i \leq m \leq B_i} \psi(a_2 + (m - a_1) \tan \varphi_i),$$

where A_i and B_i are integers satisfying $0 \leq A_i \leq B_i \leq \sqrt{2} N^{1/2}$, and where $\psi(z) = z - [z] - 1/2$ for every $z \in \mathbb{R}$. Since $\tan \varphi_i$ is badly approximable, giving

rise to good distribution of the sequence $m \tan \varphi_i$ modulo 1, the well-known result of Lerch [10] (see also Hardy and Littlewood [8, 9] and Davenport [6]) shows that

$$\sum_{A_i \leq m \leq B_i} \psi(a_2 + (m - a_1) \tan \varphi_i) \ll_{\varphi_i} \log(B_i - A_i + 2) \ll_{\varphi_i} \log N.$$

This establishes the inequality (10), and completes the proof of Lemma 2.3. $\qquad\square$

3. An Argument of Beck

To study Theorem 2, we use an elaboration of the idea of Beck as discussed in Section 8.1 of [3]. It is convenient to restrict the natural number N to be a perfect square, so that $N = M^2$ for some natural number M. This restriction can be lifted easily, in view of Lagrange's theorem that every positive integer is a sum of at most four integer squares, so that we can superimpose up to four point distributions where the number of points in each is a perfect square.

We shall consider a rescaled version of the problem, and study sets of N points in the square $[0, M]^2$. Let $k \in \mathbb{N}$ be fixed, with $k \geq 3$. We denote by $\mathcal{G}_k$ the collection of all convex polygons in $[0, M]^2$ which have at most k sides. Suppose that $\mathcal{P}$ is a set of N points in $[0, M]^2$. For every measurable subset $A \subseteq [0, M]^2$, let $Z[\mathcal{P}; A]$ denote the number of points of $\mathcal{P}$ that fall into A, and let $E[\mathcal{P}; A] = Z[\mathcal{P}; A] - \mu(A)$ denote the corresponding discrepancy. We would like to show that there exists a set $\mathcal{P}$ of N points in $[0, M]^2$ such that for every convex polygon $A \in \mathcal{G}_k$, we have

$$|E[\mathcal{P}; A]| \ll_k N^{1/4}(\log N)^{1/2}.$$

Our first step is to approximate the convex polygons in $\mathcal{G}_k$ by a special finite collection of polygons. Let $\delta = (6kM)^{-1}$, and let $\mathcal{H}_k$ denote the collection of all convex polygons in $[0, M]^2$ with at most $4k$ sides and with vertices on $(\delta\mathbb{Z})^2 \cap [0, M]^2$. It is easy to see that $|(\delta\mathbb{Z})^2 \cap [0, M]^2| = (6kN + 1)^2$, so that

$$|\mathcal{H}_k| \leq \sum_{d=3}^{4k} \binom{(6kN + 1)^2}{d} \leq c_k N^{8k},$$

where the constant c_k depends at most on k.

Lemma 3.1. *For every convex polygon $A \in \mathcal{G}_k$, there exist two convex polygons $B^+, B^- \in \mathcal{H}_k$ such that $B^- \subseteq A \subseteq B^+$ and $\mu(B^+ \setminus B^-) \leq 1$.*

Lemma 3.2. *There exists a set $\mathcal{P}$ of N points in $[0, M]^2$ such that for every convex polygon $B \in \mathcal{H}_k$, we have*

$$|E[\mathcal{P}; B]| \leq C_k N^{1/4}(\log N)^{1/2},$$

where the constant C_k depends at most on k.

Before we establish these two lemmas, we shall first complete the very short deduction of Theorem 2.

Deduction of Theorem 2. For every convex polygon $A \in \mathcal{G}_k$, it is not difficult to show that the convex polygons $B^+, B^- \in \mathcal{H}_k$ given by Lemma 3.1 satisfy the inequality

$$|E[\mathcal{P}; A]| \leq \max\{|E[\mathcal{P}; B^-]|, |E[\mathcal{P}; B^+]|\} + \mu(B^+ \setminus B^-)$$
$$\leq C_k N^{1/4}(\log N)^{1/2} + 1$$

This gives Theorem 2 immediately. $\qquad\square$

We shall establish Lemma 3.2 in Section 4, and Lemma 3.1 in Section 5.

4. Large Deviation

In this section, we establish Lemma 3.2 using a large deviation type argument. For every $\mathbf{l} = (\ell_1, \ell_2) \in \mathbb{Z}^2 \cap [0, M)^2$, let $\mathbf{q_l} \in S(\mathbf{l}) = [\ell_1, \ell_1 + 1) \times [\ell_2, \ell_2 + 1)$ be a random point uniformly distributed in $S(\mathbf{l})$ and independent of the points in the other squares, and consider the random point set

$$\widetilde{\mathcal{P}} = \{\mathbf{q_l} : \mathbf{l} \in \mathbb{Z}^2 \cap [0, M)^2\}.$$

Consider a fixed convex polygon $B \in \mathcal{H}_k$, and let

$$\mathcal{L}(B) = \{\mathbf{l} \in \mathbb{Z}^2 \cap [0, M)^2 : S(\mathbf{l}) \cap \partial B \neq \emptyset\}.$$

Then it is easy to show that

$$|\mathcal{L}(B)| \leq 4N^{1/2}.$$

For any $\mathbf{l} \in \mathcal{L}(B)$, let

$$\xi_{\mathbf{l}} = \begin{cases} 1 & \text{if } \mathbf{q_l} \in B, \\ 0 & \text{otherwise.} \end{cases}$$

Then

$$E[\widetilde{\mathcal{P}}; B] = \sum_{\mathbf{l} \in \mathcal{L}(B)} (\xi_{\mathbf{l}} - \mathsf{E}\xi_{\mathbf{l}}).$$

We now use the following large deviation type inequality due to Hoeffding; see, for example, Appendix B of Pollard [11].

Lemma 4.1. *Suppose that $\xi_1, \ldots, \xi_m$ are independent random variables such that $0 \leq \xi_i \leq 1$ for every $i = 1, \ldots, m$. Then for every $\gamma > 0$,*

$$\text{Prob}\left(\left|\sum_{i=1}^m (\xi_i - \mathsf{E}\xi_i)\right| \geq \gamma\right) \leq 2\mathrm{e}^{-2\gamma^2/m}.$$

Note that

$$m = |\mathcal{L}(B)| \leq 4N^{1/2},$$

and choose $\gamma = C_k N^{1/4}(\log N)^{1/2}$ with a sufficiently large constant C_k. Then it is easy to check that

$$\frac{\gamma^2}{m} \geq \frac{C_k^2 N^{1/2} \log N}{4N^{1/2}} = \frac{C_k^2}{4} \log N,$$

so that

$$4\mathrm{e}^{-2\gamma^2/m} \leq 4N^{-C_k^2/2} \leq c_k^{-1} N^{-8k},$$

where the last inequality is valid for all $N \geq 2$ provided that C_k is large enough in terms of k and c_k. Since

$$\tfrac{1}{2}|\mathcal{H}_k|^{-1} \geq \tfrac{1}{2}c_k^{-1}N^{-8k} \geq 2\mathrm{e}^{-2\gamma^2/m},$$

we have

$$\text{Prob}\left(|E[\widetilde{\mathcal{P}}; B]| \geq C_k N^{1/4}(\log N)^{1/2}\right) \leq \tfrac{1}{2}|\mathcal{H}_k|^{-1}.$$

If we now consider all convex polygons $B \in \mathcal{H}_k$, then the above implies

$$\text{Prob}\left(|E[\widetilde{\mathcal{P}}; B]| \geq C_k N^{1/4}(\log N)^{1/2} \text{ for some } B \in \mathcal{H}_k\right) \leq \tfrac{1}{2},$$

and so

$$\text{Prob}\left(|E[\widetilde{\mathcal{P}}; B]| \leq C_k N^{1/4}(\log N)^{1/2} \text{ for all } B \in \mathcal{H}_k\right) \geq \tfrac{1}{2}.$$

This completes the proof of Lemma 3.2.

5. Convexity

In this section, we establish Lemma 3.1 using a convexity argument. Recall that $\mathcal{G}_k$ denotes the collection of all convex polygons in $[0, M]^2$ which have at most k sides, and $\mathcal{H}_k$ denotes the collection of all convex polygons in $[0, M]^2$ with at most $4k$ sides and with vertices on $(\delta\mathbb{Z})^2 \cap [0, M]^2$, where $\delta = (6kM)^{-1}$.

For convenience, we make an *ad hoc* definition. By a δ-square, we mean a closed square of side δ and with all vertices in $(\delta\mathbb{Z})^2 \cap [0, M]^2$.

5.1. The Outer Convex Polygon B^+.
Suppose that a convex polygon $A \in \mathcal{G}_k$ is given. Corresponding to every vertex $\mathbf{v}$ of A, we shall define the set $\mathcal{O}_{\mathbf{v}}$ of "outer grid points" corresponding to $\mathbf{v}$. We distinguish two cases:

- Case 1: Suppose that $\mathbf{v} \in (\delta\mathbb{Z})^2 \cap [0, M]^2$. Then we take $\mathcal{O}_{\mathbf{v}} = \{\mathbf{v}\}$.
- Case 2: Suppose that $\mathbf{v} \notin (\delta\mathbb{Z})^2 \cap [0, M]^2$. Then we take $\mathcal{O}_{\mathbf{v}}$ to be the collection of the vertices outside A or on the boundary of A of all δ-squares that contain $\mathbf{v}$ and whose interior intersect the boundary of A.

To construct the convex polygons $B^+ \in \mathcal{H}_k$ given in Lemma 3.1, we simply let

$$B^+ = \text{ch}\left\{\bigcup \mathcal{O}_{\mathbf{v}} : \mathbf{v} \text{ is a vertex of } A\right\}$$

denote the convex hull of all the outer grid points of A. Trivially, the convex polygon B^+ has at most $4k$ sides, since A has at most k sides. The inclusion $A \subseteq B^+$ is immediate from our definition. On the other hand, we have

$$\mu(B^+ \setminus A) \leq \tfrac{1}{2}. \tag{11}$$

To see this, note that any point of $\mathcal{O}_{\mathbf{v}}$ has vertical or horizontal distance at most 2δ from the (extended) edges of A that intersect at $\mathbf{v}$. It follows that the set $B^+ \setminus A$ is contained in the union of k sets, each of area at most $2\delta M$. The inequality (11) follows immediately.

5.2. The Inner Convex Polygon B^-.
Suppose that a convex polygon $A \in \mathcal{G}_k$ is given. Here we run into some technical complications caused by the possibility of A having some vertices that are very close together. To overcome these complications, we introduce an iterative process whereby we can remove some of the vertices of A, one at a time, to obtain a smaller polygon A^*.

Start with $A_0 = A$. For each $i = 0, 1, 2, \ldots$, we remove, if possible, a vertex of the polygon A_i by taking one of the steps below, and denote by A_{i+1} the convex polygon formed with the remaining vertices:

- Option 1: Remove a vertex $\mathbf{v}$ of A_i if a δ-square containing $\mathbf{v}$ contains another vertex of A_i.
- Option 2: Remove a vertex $\mathbf{v}$ of A_i if all four vertices of every δ-square containing $\mathbf{v}$ lie outside A_i and at least one of the following two conditions is satisfied:
  - The horizontal distance from $\mathbf{v}$ to an adjacent vertex of A_i is less than the horizontal distance in the same direction from $\mathbf{v}$ to any grid point of $(\delta\mathbb{Z})^2 \cap [0, M]^2$ lying inside A_i or on the boundary of A_i.
  - The vertical distance from $\mathbf{v}$ to an adjacent vertex of A_i is less than the vertical distance in the same direction from $\mathbf{v}$ to any grid point of $(\delta\mathbb{Z})^2 \cap [0, M]^2$ lying inside A_i or on the boundary of A_i.

Note that $A_{i+1} \subseteq A_i$, and $\mu(A_i \setminus A_{i+1}) \leq \delta M$.

This iterative process stops when it is no longer possible to remove any vertex of a convex polygon under either option, and we denote by A^* the last convex polygon obtained from A by this process. Note that

$$\mu(A \setminus A^*) \leq j\delta M, \tag{12}$$

where j is the number of vertices of A removed by this process. Note that the convex polygon A^* may not be unique, and has at most $k - j$ sides.

Corresponding to every vertex $\mathbf{v}$ of A^*, we shall define the set $\mathcal{I}_{\mathbf{v}}$ of "inner grid points" corresponding to $\mathbf{v}$. We distinguish two cases:

- Case 1: Suppose that $\mathbf{v} \in (\delta\mathbb{Z})^2 \cap [0, M]^2$. Then we take $\mathcal{I}_{\mathbf{v}} = \{\mathbf{v}\}$.
- Case 2: Suppose that $\mathbf{v} \notin (\delta\mathbb{Z})^2 \cap [0, M]^2$. Let $\mathcal{F}_{\mathbf{v}}$ denote the collection of vertices inside A^* or on the boundary of A^* of all δ-squares that contain $\mathbf{v}$ and whose interior intersect the boundary of A^* – there is only one such δ-square, unless $\mathbf{v}$ lies on the boundary of two adjacent ones in which case there are precisely two. There are three possibilities:
  - If $\mathcal{F}_{\mathbf{v}} \neq \emptyset$, then we take $\mathcal{I}_{\mathbf{v}} = \mathcal{F}_{\mathbf{v}}$.
  - If $\mathcal{F}_{\mathbf{v}} = \emptyset$, and no point of the lattice $(\delta\mathbb{Z})^2 \cap [0, M]^2$ lies inside A^* or on the boundary of A^*, then we take $\mathcal{I}_{\mathbf{v}} = \emptyset$.
  - If $\mathcal{F}_{\mathbf{v}} = \emptyset$, and there are points of the lattice $(\delta\mathbb{Z})^2 \cap [0, M]^2$ that lie inside A^* or on the boundary of A^*, then for every δ-square that contains $\mathbf{v}$ and whose interior intersects the boundary of A^*, one or more of its four edges must have the following property: The edge intersects A^*, and there is a grid line of $(\delta\mathbb{Z})^2 \cap [0, M]^2$, parallel to this edge, closest to $\mathbf{v}$ but on the other side of this edge from $\mathbf{v}$, that contains points of $(\delta\mathbb{Z})^2 \cap [0, M]^2$ that lie inside A^* or on the boundary of A^*. We take $\mathcal{I}_{\mathbf{v}}$ to include all such grid points of $(\delta\mathbb{Z})^2 \cap [0, M]^2$ on these closest grid lines that lie inside A^* or on the boundary of A^*. The following is easy to prove: If the boundary of A^* crosses precisely one edge or three edges of the δ-square, then the elements of $\mathcal{I}_{\mathbf{v}}$ arising from this δ-square lie on at most one grid line. If the boundary of A^* crosses precisely two edges of the δ-square, then the elements of $\mathcal{I}_{\mathbf{v}}$ arising from this δ-square lie on at most two distinct grid lines, only one of which can contain more than one element of $\mathcal{I}_{\mathbf{v}}$. Note that the boundary of A^* cannot cross all four edges of the δ-square, as this would imply that no point of the lattice $(\delta\mathbb{Z})^2 \cap [0, M]^2$ lies inside A^* or on the boundary of A^*.

To construct the convex polygons $B^- \in \mathcal{H}_k$ given in Lemma 3.1, we simply let

$$B^- = \mathrm{ch}\left\{\bigcup \mathcal{I}_{\mathbf{v}} : \mathbf{v} \text{ is a vertex of } A^*\right\}$$

denote the convex hull of all the inner grid points of A^*, with the convention that $B^- = \emptyset$ if $\mathcal{I}_{\mathbf{v}} = \emptyset$ for every vertex $\mathbf{v}$ of A^*. Trivially, the convex polygon B^- has fewer than $4k$ sides, since A^* has at most k sides. The inclusions $B^- \subseteq A^* \subseteq A$ are immediate from our definitions. On the other hand, we have

$$\mu(A \setminus B^-) \leq \tfrac{1}{2}. \tag{13}$$

To see this, note that each vertex $\mathbf{v}$ of A^* contributes at most three vertices of B^-. Moreover, any point of $\mathcal{I}_{\mathbf{v}}$ has vertical or horizontal distance at most δ from the edges of A^* that intersect at $\mathbf{v}$. It follows that the set $A^* \setminus B^-$ is contained in the union of $k - j$ sets "along the edges", each of area at most δM, and the union of at most $2(k - j)$ triangles "near the vertices", each of area at most δM. The inequality (13) then follows at once on noting the inequality (12). The case when $B^- = \emptyset$ is trivial.

6. An Elementary Geometric Argument

In this section, we adapt the wonderfully elegant geometric argument described in Schmidt [13] to give a simple proof of Theorem 3.

Consider the circle of radius $1/2$ lying within the unit square $[0,1]^2$. Now let $k = [N^{1/3}]$, and let A denote a regular convex polygon of k sides inscribed in this circle. Elementary calculation shows that any triangle whose three vertices are one of the vertices of A and the midpoints of the two adjacent edges has area

$$\frac{1}{4}\sin^3\frac{\pi}{k}\cos\frac{\pi}{k} \geq \frac{1}{8}\left(\frac{2}{\pi}\frac{\pi}{k}\right)^3 = \frac{1}{k^3} \geq \frac{1}{N}. \tag{14}$$

Corresponding to each vertex of A, we now consider an isosceles triangle of area $1/2N$ and with its two equal sides lying on the two edges of A adjacent to this vertex. Let $B_1,\ldots,B_s$ denote those isosceles triangles which contain points of $\mathcal{P}$, and let $C_1,\ldots,C_t$ denote those isosceles triangles which do not contain points of $\mathcal{P}$. Clearly

$$D[\mathcal{P};B_i] \geq \tfrac{1}{2} \quad \text{for every } i = 1,\ldots,s,$$

and

$$D[\mathcal{P};C_j] = -\tfrac{1}{2} \quad \text{for every } j = 1,\ldots,t.$$

Furthermore, the triangles $B_1,\ldots,B_s,C_1,\ldots,C_t$ are pairwise disjoint, in view of (14) above, and $s + t = k = [N^{1/3}]$. It is also easy to see that both

$$A^+ = A \setminus (B_1 \cup \ldots \cup B_s) \quad \text{and} \quad A^- = A \setminus (C_1 \cup \ldots \cup C_t)$$

are convex polygons. But now

$$D[\mathcal{P};A^-] - D[\mathcal{P};A^+] = \sum_{i=1}^{s} D[\mathcal{P};B_i] - \sum_{j=1}^{t} D[\mathcal{P};C_j] \geq \frac{s}{2} + \frac{t}{2} = \frac{k}{2} = \frac{1}{2}[N^{1/3}].$$

It follows that

$$|D[\mathcal{P};A^-]| \geq \tfrac{1}{4}[N^{1/3}] \quad \text{or} \quad |D[\mathcal{P};A^+]| \geq \tfrac{1}{4}[N^{1/3}],$$

and this completes the proof of Theorem 3.

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