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DISCREPANCY FOR RANDOMIZED RIEMANN SUMS

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ABSTRACT. Given a finite sequence $U_N = \{u_1, \ldots, u_N\}$ of points contained in the d-dimensional unit torus, we consider the L^2 discrepancy between the integral of a given function and the Riemann sums with respect to translations of U_N. We show that with positive probability, the L^2 discrepancy of other sequences close to U_N in a certain sense preserves the order of decay of the discrepancy of U_N. We also study the role of the regularity of the given function.

1. INTRODUCTION

Let $N \in \mathbb{N}$ be a given large number, let $U_N = \{u_1, \ldots, u_N\}$ be a distribution of N points in the unit cube $[-\frac{1}{2}, \frac{1}{2})^d$, treated as the torus $\mathbb{T}^d$, and let f be a real function on $\mathbb{T}^d$. Suppose that for suitable choices of U_N and f, the Riemann sums

$$\frac{1}{N} \sum_{j=1}^{N} f(u_j - x)$$

are, after an L^2 average on the variable $x \in \mathbb{T}^d$, good approximations of the integral

$$\int_{\mathbb{T}^d} f(s) \, \mathrm{d}s.$$

What corresponding statement can we make concerning those sequences *close* to the sequence U_N? Do such sequences mostly share the same good behavior?

2. A RANDOMIZATION ARGUMENT

In order to start discussing these questions, we introduce the following randomization of U_N; see [3, 6] and also [8, 9]. Let $\mathrm{d}\mu$ denote a probability measure on $\mathbb{T}^d$. For every $j = 1, \ldots, N$, let $\mathrm{d}\mu_j$ denote the measure obtained after translating $\mathrm{d}\mu$ by u_j. More precisely, for any integrable function g on $\mathbb{T}^d$, we have

$$\int_{\mathbb{T}^d} g(t) \, \mathrm{d}\mu_j = \int_{\mathbb{T}^d} g(t - u_j) \, \mathrm{d}\mu.$$

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Let $\mathrm{d}t$ denote the Lebesgue measure on $\mathbb{T}^d$. For every sequence $V_N = \{v_1, \ldots, v_N\}$ in $\mathbb{T}^d$ and every function $f \in L^2(\mathbb{T}^d, \mathrm{d}t)$, we introduce, for every $t \in \mathbb{T}^d$, the discrepancy

$$D(t, V_N) \stackrel{\text{def}}{=} \frac{1}{N} \sum_{j=1}^{N} f(v_j - t) - \int_{\mathbb{T}^d} f(s)\, \mathrm{d}s.$$

Observe that $D(\cdot, V_N)$ is a periodic function with Fourier series

$$\sum_{0 \neq k \in \mathbb{Z}^d} \left(\frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{-2\pi \mathrm{i} k \cdot v_j} \right) \overline{\widehat{f}(k)} \mathrm{e}^{2\pi \mathrm{i} k \cdot t},$$

and the Parseval identity yields

$$D^2(V_N) \stackrel{\text{def}}{=} \|D(\cdot, V_N)\|^2_{L^2(\mathbb{T}^d, \mathrm{d}t)} = \sum_{0 \neq k \in \mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{2\pi \mathrm{i} k \cdot v_j} \right|^2 |\widehat{f}(k)|^2.$$

We now average $D(V_N)$ in $L^2(\mathbb{T}^d, \mathrm{d}\mu_j)$ for every $j = 1, \ldots, N$, and consider

$$\mathfrak{D}_{\mathrm{d}\mu}(U_N) \stackrel{\text{def}}{=} \left(\int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} D^2(V_N)\, \mathrm{d}\mu_1(v_1) \ldots \mathrm{d}\mu_N(v_N) \right)^{1/2}.$$

In this paper we study the relation between $\mathfrak{D}_{\mathrm{d}\mu}(U_N)$ and $D(U_N)$. In the case $N = M^d$, where $M \in \mathbb{N}$, and

$$U_N = \frac{1}{M} \mathbb{Z}^d \cap \left[-\frac{1}{2}, \frac{1}{2} \right)^d, \tag{1}$$

the above quantities were studied in relation to the sharpness of a result of Beck [1] and of Montgomery [10] on irregularities of distribution; see Remark 3 below. In [6] two of the authors compared the quantities $D(U_N)$ and $\mathfrak{D}_{\mathrm{d}\mu}(U_N)$ in the case (1) and when f is the characteristic function of a ball. Here we study the problem in our more general setting, and we are mainly interested in whether the inequality

$$\mathfrak{D}_{\mathrm{d}\mu}(U_N) \leqslant c\, D(U_N) \tag{2}$$

holds.

Throughout this paper, the letters $c, C, \ldots$ will denote positive constants, possibly depending on f but independent of N, and which may change from one step to the next. On the other hand, different letters $B, \kappa, \ldots$ will denote constants which will not change throughout the paper.

3. An Explicit Formula

We first use a slight modification of an argument in [6] to obtain an explicit formula for $\mathfrak{D}_{\mathrm{d}\mu}(U_N)$. We have

$$
\begin{aligned}
(3)\quad & \mathfrak{D}^2_{\mathrm{d}\mu}(U_N) \\
&= \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} \sum_{0\neq k\in\mathbb{Z}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \mathrm{e}^{2\pi\mathrm{i}k\cdot v_j} \right|^2 |\widehat{f}(k)|^2 \,\mathrm{d}\mu_1(v_1)\ldots\mathrm{d}\mu_N(v_N) \\
&= \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{f}(k)|^2 \left(\frac{1}{N} + \frac{1}{N^2} \sum_{\substack{j,\ell=1\\ j\neq\ell}}^{N} \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathrm{e}^{2\pi\mathrm{i}k\cdot v_j}\mathrm{e}^{-2\pi\mathrm{i}k\cdot v_\ell}\,\mathrm{d}\mu_j(v_j)\mathrm{d}\mu_\ell(v_\ell) \right) \\
&= \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{f}(k)|^2 \\
&\quad\times \left(\frac{1}{N} + \frac{1}{N^2} \sum_{\substack{j,\ell=1\\ j\neq\ell}}^{N} \mathrm{e}^{2\pi\mathrm{i}k\cdot(u_\ell-u_j)} \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \mathrm{e}^{2\pi\mathrm{i}k\cdot v_j}\mathrm{e}^{-2\pi\mathrm{i}k\cdot v_\ell}\,\mathrm{d}\mu(v_j)\mathrm{d}\mu(v_\ell) \right) \\
&= \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{f}(k)|^2 \left(\frac{1}{N} + |\widehat{\mu}(k)|^2 \left(\left| \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{2\pi\mathrm{i}k\cdot u_j} \right|^2 - \frac{1}{N} \right) \right) \\
&= \frac{1}{N} \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{f}(k)|^2 (1-|\widehat{\mu}(k)|^2) + \sum_{0\neq k\in\mathbb{Z}^d} |\widehat{f}(k)|^2 |\widehat{\mu}(k)|^2 \left| \frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{2\pi\mathrm{i}k\cdot u_j} \right|^2 \\
&= \frac{1}{N}\left(\|f\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)} - \|f*\mathrm{d}\mu\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)} \right) + \|D(\cdot,U_N)*\mathrm{d}\mu\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)}.
\end{aligned}
$$

There are two natural extremal measures. The first one is $\mathrm{d}\mu = \delta_0$, the Dirac measure centred at 0. In this case, we have

$$\mathfrak{D}_{\delta_0}(U_N) = D(U_N).$$

On the other hand, when $\mathrm{d}\mu = \mathrm{d}t$, we have

$$\mathfrak{D}^2_{\mathrm{d}t}(U_N) = \frac{1}{N}\left(\|f\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)} - \left| \int_{\mathbb{T}^d} f(t)\,\mathrm{d}t \right|^2 \right),$$

the classical Monte-Carlo error.

Note that if $ND^2(U_N) \geqslant c$, then $\mathfrak{D}_{\mathrm{d}t}(U_N) \leqslant c_1 D(U_N)$, and (2) follows easily.

Another very peculiar case is when $D(U_N) = 0$. We observe that in general this does not imply $\mathfrak{D}_{\mathrm{d}\mu}(U_N) = 0$, so that (2) does not hold. Indeed, let U_N be given by (1). Then

$$\frac{1}{N}\sum_{j=1}^{N} \mathrm{e}^{2\pi\mathrm{i}k\cdot u_j} = \begin{cases} 1 & \text{if } k \in M\mathbb{Z}^d, \\ 0 & \text{otherwise.} \end{cases} \tag{4}$$

Now choose $f(t) = \exp(2\pi \mathrm{i} k_0 \cdot t)$ for some $k_0 \in \mathbb{Z}^d \setminus M\mathbb{Z}^d$. Then $D(U_N) = 0$. On the other hand, it follows from (3) that

$$\mathfrak{D}^2_{\mathrm{d}\mu}(U_N) = \frac{1}{N}(1 - |\widehat{\mu}(k_0)|^2) \neq 0$$

whenever $|\widehat{\mu}(k_0)| \neq 1$, which is easily fulfilled, particularly by several measures with small support around the origin.

Hence, throughout the paper, we will be interested only in the case when

$$0 < D(U_N) < N^{-1/2}.$$

Let $0 < \varepsilon_N \leqslant 1$. For every probability measure $\mathrm{d}\mu$ supported on the unit cube $[-\frac{1}{2}, \frac{1}{2})^d$, let $\mathrm{d}\mu^{(N)}$ denote the probability measure defined by

$$\int_{\mathbb{R}^d} g(\xi) \, \mathrm{d}\mu^{(N)}(\xi) = \int_{\mathbb{R}^d} g(\varepsilon_N \xi) \, \mathrm{d}\mu(\xi). \tag{5}$$

Then $\mathrm{d}\mu^{(N)}$ is supported on the subcube $[-\frac{1}{2}\varepsilon_N, \frac{1}{2}\varepsilon_N)^d$, and can be regarded as a measure on $\mathbb{T}^d$.

4. Main Result

We first state our main result.

Theorem 1. *Let $f \in L^2(\mathbb{T}^d, \mathrm{d}t)$ and let $U_N = \{u_1, \dots, u_N\}$ be a distribution of N points in the cube $[-\frac{1}{2}, \frac{1}{2})^d$. Assume that $0 < D(U_N) < N^{-1/2}$. Let $\mathrm{d}\mu$ be a non-Dirac probability measure on $\mathbb{T}^d$, let $\mathrm{d}\mu^{(N)}$ be defined by (5) with $0 < \varepsilon_N \leqslant 1$, and let*

$$\eta_N = \begin{cases} \varepsilon_N^{2\alpha} & \text{if } \alpha < 1, \\ \varepsilon_N^2 \log(1 + \varepsilon_N^{-1}) & \text{if } \alpha = 1, \\ \varepsilon_N^2 & \text{if } \alpha > 1. \end{cases}$$

(i) *If for some $\alpha > 0$ and for every $\rho > 1$ we have*

$$\sum_{\rho \leqslant |k| < 2\rho} |\widehat{f}(k)|^2 \leqslant c\, \rho^{-2\alpha}, \tag{6}$$

then

$$\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) \leqslant c\, \eta_N N^{-1} + D^2(U_N). \tag{7}$$

(ii) *If there exists an open cone[1] $\Omega \subseteq \mathbb{R}^d$ such that for every subcone $\Gamma \subseteq \Omega$,*

$$\liminf_{\rho \to +\infty} \rho^{2\alpha} \sum_{\substack{k \in \Gamma \\ \rho \leqslant |k| < 2\rho}} |\widehat{f}(k)|^2 > 0, \tag{8}$$

then there exist positive constants $\Delta \leqslant 1$ and c such that if $\varepsilon_N \leqslant \Delta$, then

$$\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) \geqslant c\, \eta_N N^{-1}.$$

The following corollary shows that, in some sense, good sequences are never alone. Indeed we give conditions on ε_N that will ensure that $\mathfrak{D}_{\mathrm{d}\mu^{(N)}}(U_N)$ and $D(U_N)$ are comparable.

[1] In this paper every cone starts from the origin.

Corollary 2. *Let f, U_N and $\mathrm{d}\mu$ be as given in Theorem 1.*

(i) *Let f be as given in part (i) of Theorem 1 and let*

$$\varepsilon_N \leqslant \begin{cases} (N^{1/2}D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\ \beta_N & \text{if } \alpha = 1, \\ N^{1/2}D(U_N) & \text{if } \alpha > 1, \end{cases} \tag{9}$$

where β_N satisfies $\beta_N^2 \log(1+\beta_N^{-1}) = ND^2(U_N)$. Then

$$\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) \leqslant c\, D^2(U_N).$$

(ii) *Let f and Δ be as given in part (ii) of Theorem 1 and let $\kappa > 0$. Then there exists $c > 0$ such that whenever*

$$\Delta \geqslant \varepsilon_N \geqslant \begin{cases} \kappa(N^{1/2}D(U_N))^{1/\alpha} & \text{if } \alpha < 1, \\ \kappa\beta_N & \text{if } \alpha = 1, \\ \kappa N^{1/2}D(U_N) & \text{if } \alpha > 1, \end{cases} \tag{10}$$

we have

$$\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) \geqslant c\, D^2(U_N).$$

Remark 3. Consider the particular case when $f = \chi_A$, the characteristic function of a convex body $A \subseteq [-\frac{1}{2}, \frac{1}{2})^d$. Then (6) holds with $\alpha = \frac{1}{2}$. Let $\varepsilon_N = \Delta\, ND^2(U_N)$. Then

$$\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) \leqslant c\, D^2(U_N).$$

If furthermore the boundary of A is smooth and has positive Gaussian curvature then (8) holds with $\alpha = \frac{1}{2}$; see, for instance, [7]. We then have

$$\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) \geqslant c\, D^2(U_N).$$

We recall that if A is rotated and contracted, then a result of Beck [1] and of Montgomery [10] says that

$$\int_{SO(d)} \int_0^1 \int_{\mathbb{T}^d} \left| \frac{1}{N} \sum_{j=1}^{N} \chi_{\sigma(rA)}(u_j - t) - r^d |A| \right|^2 \mathrm{d}t\, \mathrm{d}r\, \mathrm{d}\sigma \geqslant c\, N^{-1-1/d}$$

for every choice of the point set distribution U_N; see also [2, 4, 5]. We also recall that this is not true if the contraction is omitted; see [12, Theorem 3.1].

5. Decay of the Fourier Coefficients

The assumption (6) concerns the decay of the Fourier coefficients of f. This behavior can be naturally related to the smoothness of the function f as follows. Let $f \in L^2(\mathbb{T}^d)$, define $\Delta_h f(x) = f(x+h) - f(x)$ and, for every integer $\ell \geqslant 1$, write $\Delta_h^\ell f = \Delta_h \Delta_h^{\ell-1} f$. Let $\alpha > 0$. We say that f belongs to the Nikol'skiĭ space $H_2^\alpha(\mathbb{T}^d)$ if there exists $c > 0$ such that

$$\left(\int_{\mathbb{T}^d} |\Delta_h^\ell f(x)|^2 \,\mathrm{d}x \right)^{1/2} \leqslant c\, |h|^\alpha$$

for some $\ell \geqslant 1$; see [11, Section 4.3.3].

Proposition 4. *Let $f \in H_2^\alpha(\mathbb{T}^d)$. Then (6) holds.*

Proof. Since $\widehat{\Delta_h f}(k) = (\mathrm{e}^{2\pi \mathrm{i} k\cdot h} - 1)\widehat{f}(k)$, we have $\widehat{\Delta_h^\ell f}(k) = (\mathrm{e}^{2\pi \mathrm{i} k\cdot h} - 1)^\ell \widehat{f}(k)$. Let $h = (1/10\rho, 0, \ldots, 0)$ and $\Gamma = \{k \in \mathbb{Z}^d : k_1^2 \geqslant k_2^2 + \ldots + k_d^2\}$. Observe that when $k \in \Gamma$ and $\rho \leqslant |k| \leqslant 2\rho$, we have $|\mathrm{e}^{2\pi \mathrm{i} k\cdot h} - 1| \geqslant c$. Therefore

$$\begin{aligned}\sum_{\substack{k\in\Gamma\\ \rho\leqslant|k|<2\rho}} |\widehat{f}(k)|^2 &\leqslant c \sum_{\substack{k\in\Gamma\\ \rho\leqslant|k|<2\rho}} |(\mathrm{e}^{2\pi \mathrm{i} k\cdot h} - 1)^\ell|^2 |\widehat{f}(k)|^2 \leqslant c \sum_{k\in\mathbb{Z}^d} |\widehat{\Delta_h^\ell f}(k)|^2 \\ &= c \int_{\mathbb{T}^d} |\Delta_h^\ell f(x)|^2 \,\mathrm{d}x \leqslant c\,|h|^{2\alpha} = c\,\rho^{-2\alpha}.\end{aligned}$$

Note here that h is tailored on Γ. Since we can cover $\mathbb{Z}^d$ with a finite number of cones, the proposition follows from the above argument applied to different choices of h. □

We begin the proof of Theorem 1 with a technical lemma.

Lemma 5. *Let* $\mathrm{d}\nu$ *be a probability measure supported on* $[-\frac{1}{2}, \frac{1}{2})^d$. *Then either*

(i) $\mathrm{d}\nu$ *is the Dirac measure* δ_{t_0} *at a point* $t_0 \in [-\frac{1}{2}, \frac{1}{2})^d$; *or*
(ii) $1 - |\widehat{\nu}(\xi)|^2 = O(|\xi|^2)$ *as* $\xi \to 0$, *and any open cone in* $\mathbb{R}^d$ *contains an open subcone* Γ *such that* $1 - |\widehat{\nu}(\xi)|^2 \geqslant c\,|\xi|^2$ *for small* $\xi \in \Gamma$.

Proof. Since $\mathrm{d}\nu$ is compactly supported, its Fourier transform $\widehat{\nu}$ is smooth and has Taylor expansion

$$\widehat{\nu}(\xi) = 1 + \nabla\widehat{\nu}(0)\cdot\xi + \tfrac{1}{2} H_{\widehat{\nu}}(0)\xi\cdot\xi + o(|\xi|^2),$$

and so

$$1 - |\widehat{\nu}(\xi)|^2 = 1 - \widehat{\nu}(\xi)\widehat{\nu}(-\xi) = (\nabla\widehat{\nu}(0)\cdot\xi)^2 - H_{\widehat{\nu}}(0)\xi\cdot\xi + o(|\xi|^2) = O(|\xi|^2).$$

Let $F(\xi) = (\nabla\widehat{\nu}(0)\cdot\xi)^2 - H_{\widehat{\nu}}(0)\xi\cdot\xi$, and assume that F does not vanish identically. Let $\Sigma_{d-1} = \{\xi \in \mathbb{R}^d : |\xi| = 1\}$. Since F is a polynomial, it cannot vanish on an open set and therefore $\{\xi \in \Sigma_{d-1} : F(\xi) = 0\}$ has empty interior in Σ_{d-1}. Since F is homogeneous and continuous, it follows that for every open cone in $\mathbb{R}^d$, we can find an open subcone Γ such that $|F(\xi)| \geqslant c|\xi|^2$ for $\xi \in \Gamma$. Therefore $1 - |\widehat{\nu}(\xi)|^2 \geqslant c|\xi|^2$ for small $\xi \in \Gamma$.

Assume now that $F \equiv 0$. Observe that

$$\frac{\partial\widehat{\nu}}{\partial\xi_j}(0) = -2\pi\mathrm{i}\int_{\mathbb{T}^d} x_j \,\mathrm{d}\nu(x)$$

and

$$\frac{\partial^2\widehat{\nu}}{\partial\xi_j\xi_\ell}(0) = -4\pi^2\int_{\mathbb{T}^d} x_j x_\ell \,\mathrm{d}\nu(x).$$

Then

$$\nabla\widehat{\nu}(0)\cdot\xi = -2\pi\mathrm{i}\int_{\mathbb{T}^d} (x\cdot\xi)\,\mathrm{d}\nu(x)$$

and

$$H_{\widehat{\nu}}(0)\xi\cdot\xi = -4\pi^2\sum_{i,j}\int_{\mathbb{T}^d} \xi_j\xi_\ell x_j x_\ell \,\mathrm{d}\nu(x) = -4\pi^2\int_{\mathbb{T}^d} (\xi\cdot x)^2\,\mathrm{d}\nu(x).$$

Hence

$$0 = (\nabla\widehat{\nu}(0)\cdot\xi)^2 - H_{\widehat{\nu}}(0)\xi\cdot\xi = -4\pi^2\left(\int_{\mathbb{T}^d}(x\cdot\xi)\,\mathrm{d}\nu(x)\right)^2 + 4\pi^2\int_{\mathbb{T}^d}(\xi\cdot x)^2\,\mathrm{d}\nu(x)$$
$$= 4\pi^2\int_{\mathbb{T}^d}\left(x\cdot\xi - \int_{\mathbb{T}^d}(t\cdot\xi)\,\mathrm{d}\nu(t)\right)^2\mathrm{d}\nu(x).$$

Let

$$t_0 = \int_{\mathbb{T}^d} t\,\mathrm{d}\nu(t).$$

Since $\mathrm{d}\nu(x)$ is positive, it follows that for every fixed ξ, we have

$$\nu(\{x : x\cdot\xi - \xi\cdot t_0 \neq 0\}) = 0.$$

Since ξ is arbitrary, we conclude that $\nu(\{x : x - t_0 \neq 0\}) = 0$, so that $\mathrm{d}\nu$ is supported at t_0. Since $\mathrm{d}\nu$ is a probability measure, we have $\mathrm{d}\nu = \delta_{t_0}$. $\square$

6. Proof of Theorem 1

By Lemma 5, we have

$$1 - |\widehat{\mu^{(N)}}(k)|^2 = 1 - |\widehat{\mu}(\varepsilon_N k)|^2 = O(\varepsilon_N^2|k|^2).$$

As $\mathrm{d}\mu$ is a probability measure, we have

$$0 \leqslant 1 - |\widehat{\mu^{(N)}}(k)|^2 \leqslant \min\{1, c\,\varepsilon_N^2|k|^2\}.$$

By (6), we have

$$\sum_{k\in\mathbb{Z}^d}|\widehat{f}(k)|^2(1-|\widehat{\mu^{(N)}}(k)|^2) \leqslant \sum_{k\in\mathbb{Z}^d}|\widehat{f}(k)|^2\min\{1, c\,\varepsilon_N^2|k|^2\}$$
$$\leqslant \sum_{j=0}^{+\infty}\min\{1, c\,\varepsilon_N^2 2^{2j}\}\sum_{2^j\leqslant|k|<2^{j+1}}|\widehat{f}(k)|^2 \leqslant c\sum_{j=0}^{+\infty}\min\{1,\varepsilon_N^2 2^{2j}\}2^{-2j\alpha}$$
$$\leqslant c\,\varepsilon_N^2\sum_{2^j<\varepsilon_N^{-1}}2^{(2-2\alpha)j} + c\sum_{2^j>\varepsilon_N^{-1}}2^{-2j\alpha}. \tag{11}$$

There are three cases. If $\alpha < 1$, we have

$$\sum_{0\neq k\in\mathbb{Z}^d}|\widehat{f}(k)|^2(1-|\widehat{\mu^{(N)}}(k)|^2) \leqslant c\,\varepsilon_N^{2\alpha}.$$

If $\alpha = 1$, we have

$$\sum_{0\neq k\in\mathbb{Z}^d}|\widehat{f}(k)|^2(1-|\widehat{\mu^{(N)}}(k)|^2) \leqslant c\,\varepsilon_N^2\log(1+\varepsilon_N^{-1}).$$

If $\alpha > 1$, we have

$$\sum_{0\neq k\in\mathbb{Z}^d}|\widehat{f}(k)|^2(1-|\widehat{\mu^{(N)}}(k)|^2) \leqslant c\,\varepsilon_N^2.$$

Since $\mathrm{d}\mu$ is a probability measure, we have

$$\|D(\cdot, U_N) * \mathrm{d}\mu\|_{L^2(\mathbb{T}^d,\mathrm{d}t)} \leqslant D(U_N). \tag{12}$$

In view of (11) and (12), the inequality (7) follows from (3).

Let us now prove (ii). By Lemma 5 there exists a subcone $\Gamma \subset \Omega$ such that $1 - |\widehat{\mu}(\xi)|^2 \geqslant m_1|\xi|^2$ for $|\xi| \leqslant m_2$, $\xi \in \Gamma$. By (8) there exist m_3 and m_4 such that for $\rho \geqslant m_3$ we have

$$\sum_{\substack{k\in\Gamma \\ \rho\leqslant|k|<2\rho}} |\widehat{f}(k)|^2 \geqslant m_4\rho^{-2\alpha}.$$

Thus, for $\varepsilon_N < \min\{m_2/4m_3, 1\}$, we have

$$\begin{aligned}
\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) &\geqslant \frac{1}{N}\left(\|f\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)} - \|f * \mathrm{d}\mu^{(N)}\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)}\right) \\
&= \frac{1}{N}\sum_{0\neq k\in\mathbb{Z}^d} |\widehat{f}(k)|^2(1 - |\widehat{\mu}(\varepsilon_N k)|^2) \\
&\geqslant \frac{1}{N}\sum_{m_3\leqslant 2^j\leqslant\frac{1}{2}m_2\varepsilon_N^{-1}}\ \sum_{\substack{k\in\Gamma \\ 2^j\leqslant|k|<2^{j+1}}} |\widehat{f}(k)|^2(1 - |\widehat{\mu}(\varepsilon_N k)|^2) \\
&\geqslant \frac{\varepsilon_N^2}{N} m_1 m_4 \sum_{m_3\leqslant 2^j\leqslant\frac{1}{2}m_2\varepsilon_N^{-1}} 2^{-2j\alpha}2^{2j} \geqslant c\,\eta_N N^{-1}.
\end{aligned}$$

This completes the proof of Theorem 1.

Remark 6. The estimates from below for $\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N)$ contained in Theorem 1 and Corollary 2 depend on suitable estimates for the first term

$$\frac{1}{N}\left(\|f\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)} - \|f * \mathrm{d}\mu^{(N)}\|^2_{L^2(\mathbb{T}^d,\mathrm{d}t)}\right)$$

in (3). We observe that in our setting the second term may vanish even in rather natural examples. Indeed, let

$$f(x) = \sum_{k\neq 0} \frac{1}{|k|^\gamma} \mathrm{e}^{2\pi \mathrm{i} kx}$$

for some $\gamma > d/2 + 1$. One can easily check that (8) holds with $\alpha = \gamma - d/2$. Let U_N be as in (1) and μ be the (normalized) Lebesgue measure restricted to $[-\frac{1}{2}, \frac{1}{2})^d$, so that, taking $\varepsilon_N = 1/M$, we have

$$\widehat{\mu^{(N)}}(k) = N\prod_{j=1}^d \frac{\sin(\pi k_j/M)}{\pi k_j}.$$

By (4) we have

$$D^2(U_N) = \sum_{k\neq 0} |\widehat{f}(Mk)|^2 = \frac{1}{M^{2\gamma}}\sum_{k\neq 0}\frac{1}{|k|^{2\gamma}} = \frac{c_\gamma}{M^{2\gamma}}$$

and

$$\|D(\cdot, U_N) * \mathrm{d}\mu^{(N)}\|_{L^2(\mathbb{T}^d,\mathrm{d}t)} = \sum_{k\neq 0} |\widehat{f}(Mk)|^2|\widehat{\mu^{(N)}}(Mk)|^2 = 0.$$

On the other hand observe that, for large N,

$$\varepsilon_N = \frac{1}{M} \geqslant c_\gamma N^{\frac{1}{2}} D(U_N) = c_\gamma M^{d/2-\gamma}$$

and therefore we can apply part (ii) of Corollary 2 and obtain the inequality $\mathfrak{D}^2_{\mathrm{d}\mu^{(N)}}(U_N) \geqslant c\,D^2(U_N)$.

7. Conclusion

Let $\mathrm{d}\mu^{\otimes}$ be defined on $(\mathbb{T}^d)^N$ by

$$\int_{(\mathbb{T}^d)^N} \varphi \, \mathrm{d}\mu^{\otimes} = \int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} \varphi(v_1 - u_1, \ldots, v_N - u_N) \, \mathrm{d}\mu^{(N)}(v_1) \ldots \mathrm{d}\mu^{(N)}(v_N).$$

We can now state and prove the result introduced in the abstract.

Corollary 7. *Let f, U_N and $\mathrm{d}\mu$ be as given in Corollary 2.*

(i) *Let f and ε_N be as given in part (i) of Corollary 2. Then for every λ satisfying $0 < \lambda < 1$, there exists a constant $c_\lambda > 0$, independent of U_N and such that $\mathrm{d}\mu^{\otimes}(\{V_N : D(V_N) \leqslant c_\lambda D(U_N)\}) \geqslant \lambda$.*

(ii) *Let f, Δ and ε_N be as given in part (ii) of Corollary 2. Then for a suitable constant $c > 0$, we have $\mathrm{d}\mu^{\otimes}(\{V_N : D(V_N) \geqslant cD(U_N)\}) > 0$.*

Proof. If (9) holds, then Corollary 2 gives

$$\int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} D^2(V_N) \, \mathrm{d}\mu^{\otimes}(V_N) \leqslant c \, D^2(U_N).$$

By the Chebyshev inequality, we have

$$\mathrm{d}\mu^{\otimes}(\{V_N : D(V_N) > c_\lambda D(U_N)\}) \leqslant \frac{c}{c_\lambda^2},$$

and so

$$\mathrm{d}\mu^{\otimes}(\{V_N : D(V_N) \leqslant c_\lambda D(U_N)\}) \geqslant 1 - \frac{c}{c_\lambda^2}.$$

A suitable choice of c_λ completes the proof of part (i). If (10) and (8) hold, then Corollary 2 gives

$$\int_{\mathbb{T}^d} \cdots \int_{\mathbb{T}^d} D^2(V_N) \, \mathrm{d}\mu^{\otimes}(V_N) \geqslant c \, D^2(U_N)$$

which easily implies $\mathrm{d}\mu^{\otimes}(\{V_N : D(V_N) \geqslant cD(U_N)\}) > 0$. □

References

1. J. Beck, *Irregularities of distribution I*, Acta Math. **159** (1987), 1–49.
2. J. Beck and W.W.L. Chen, *Irregularities of Distribution*, Cambridge Tracts in Mathematics, vol. 89, Cambridge University Press, 1987.
3. J. Beck and W.W.L. Chen, *Note on irregularities of distribution II*, Proc. London Math. Soc. **61** (1990), 251–272.
4. L. Brandolini, L. Colzani, and G. Travaglini, *Average decay of Fourier transforms and integer points in polyhedra*, Ark. Mat. **35** (1997), 253–275.
5. L. Brandolini, M. Rigoli, and G. Travaglini, *Average decay of Fourier transforms and geometry of convex sets*, Rev. Mat. Iberoamericana **14** (1998), 519–560.
6. W.W.L. Chen and G. Travaglini, *Deterministic and probabilistic discrepancies*, Ark. Mat. (to appear).
7. C.S. Herz, *Fourier transforms related to convex sets*, Ann. of Math. **75** (1962), 81–92.
8. F.J. Hickernell and H. Woźniakowski, *The price of pessimism for multidimensional quadrature*, J. Complexity (17) 2001, 625–659.
9. D.P. Mitchell, *Consequences of stratified samplings in graphics*, SIGGRAPH 96: Proceedings of the 23rd Annual Conference on Computer Graphics and Interactive Techniques, Association for Computing Machinery, 1996, pp. 277–280.
10. H.L. Montgomery, *Ten Lectures on the Interface between Harmonic Analysis and Analytic Number Theory*, CBMS Regional Conference Series in Mathematics, vol. 84, Amer. Math. Soc., Providence, RI, 1994.

11. S.M. Nikol'skiĭ, *Approximation of Functions of Several Variables and Imbedding Theorems*, Springer-Verlag, 1975.
12. L. Parnovski and A.V. Sobolev, *On the Bethe–Sommerfeld conjecture for the polyharmonic operator*, Duke Math. J. **107** (2001), 209–223.

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