A NOTE ON THE KRONECKER–WEYL EQUIDISTRIBUTION THEOREM

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Abstract. We study the relationship between the discrete and the continuous versions of the Kronecker–Weyl equidistribution theorem, as well as their possible extension to manifolds in higher dimensions. We also investigate a way to deduce in some limited way uniformity results in higher dimension from results in lower dimension.

1. Introduction

There are two versions of the classical Kronecker–Weyl equidistribution theorem. Let d be a fixed positive integer. If $v_1, \ldots, v_d, 1$ are linearly independent over $\mathbb{Q}$, then the vector $\mathbf{v} = (v_1, \ldots, v_d) \in \mathbb{R}^d$ is called a Kronecker vector, and the vector $\mathbf{v}^* = (v_1, \ldots, v_d, 1) \in \mathbb{R}^{d+1}$ is called a Kronecker direction. The continuous version concerns the distribution of half-infinite geodesics with Kronecker directions in the unit torus $[0,1)^{d+1}$, and much of this monograph concerns extensions of this version from the unit torus $[0,1)^2$ to arbitrary finite polysquare translation surfaces.

On the other hand, the discrete version of the Kronecker–Weyl equidistribution theorem concerns the distribution of half-infinite Kronecker sequences in the unit torus $[0,1)^d$. A natural first question is then to attempt to extend this version from the unit torus $[0,1)^2$ to arbitrary finite polysquare translation surfaces.

In general, for any fixed positive integer d, we consider the continuous problem concerning the distribution of half-infinite geodesics with Kronecker directions in finite polycube translation manifolds in $d+1$ dimensions, as well as the discrete problem of the distribution of half-infinite Kronecker sequences in finite polycube translation manifolds in d dimensions. The following are natural questions:

Question 1. *Is it true that, for any integer $d \geqslant 1$, any half-infinite geodesic with a Kronecker direction in a finite polycube translation manifold in $d+1$ dimensions is uniformly distributed? If not, then under what condition can we guarantee uniform distribution?*

Question 2. *Is it true that, for any integer $d \geqslant 2$, any half-infinite Kronecker sequence in a finite polycube translation manifold in d dimensions is uniformly distributed? If not, then under what condition can we guarantee uniform distribution?*

Question 3. *The classical Kronecker–Weyl equidistribution theorem on the unit torus has some time-quantitative extensions with explicit error terms. Under what conditions can we establish time-quantitative uniformity in these more general settings?*

For Question 1, the Gutkin–Veech theorem [4, 5] answers the special case $d = 1$ in the affirmative. However, for $d = 2$, we are currently not able to establish uniformity results for half-infinite geodesics in a general finite polycube translation 3-manifold.

In this paper, we study the relationship between the discrete and the continuous versions of possible non-integrable analogues of the Kronecker–Weyl equidistribution

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theorem concerning finite polysquare translation surfaces and some related finite polycube translation 3-manifolds. In Section 2, we investigate a relationship between the discrete version and the continuous version in the special case $d = 2$. Theorem 2, a by-product of this study, gives an affirmative answer to the special case $d = 2$ of Question 2. Then in Section 3, we develop a way to step up the problem by one dimension, and this leads to various infinite classes of polycube translation manifolds where half-infinite Kronecker sequences and half-infinite geodesics with Kronecker directions are uniformly distributed.

2. A simple equivalence principle

Before we go any further, it is appropriate that we understand what we mean by a Kronecker sequence on a finite polysquare translation surface $\mathcal{P}$. To properly define a half-infinite Kronecker sequence $\mathbf{v}_0 + j\mathbf{v}$, $j = 0, 1, 2, 3, \ldots,$ with starting point $\mathbf{v}_0$ and step vector $\mathbf{v}$ on a polysquare translation surface $\mathcal{P}$, we need a supporting half-infinite geodesic $\mathcal{L}(t)$, $t \geqslant 0$, with $\mathcal{L}(0) = \mathbf{v}_0$ on $\mathcal{P}$ and a time step $g \in \mathbb{R}$, such that the finite geodesic segment $\mathcal{L}(t)$, $0 \leqslant t \leqslant g$, is in the direction of the step vector $\mathbf{v}$ and has length equal to $|\mathbf{v}|$. Then $\mathbf{v}_0 + j\mathbf{v} = \mathcal{L}(jg)$, $j = 0, 1, 2, 3, \ldots.$ Clearly the Kronecker sequence is half-infinite if and only if $\mathbf{v}_0$ is a non-pathological starting point of a geodesic with direction $\mathbf{v}$ on $\mathcal{P}$.

Suppose that $\mathcal{P}$ is a polysquare translation surface with s atomic squares, and that $\mathcal{M} = \mathcal{P} \times [0, 1)$ denotes the associated polycube translation 3-manifold which is the cartesian product of $\mathcal{P}$ and the unit torus $[0, 1)$.

A vector $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is said to be a Kronecker vector if $v_1, v_2, 1$ are linearly independent over $\mathbb{Q}$. We are interested in the distribution of a Kronecker sequence $\mathbf{v}_0 + j\mathbf{v}$, $j = 0, 1, 2, 3, \ldots,$ with starting point $\mathbf{v}_0$ on the polysquare translation surface $\mathcal{P}$.

A vector $\mathbf{v}^* = (v_1, v_2, 1) \in \mathbb{R}^3$ is said to be a Kronecker direction if $v_1, v_2, 1$ are linearly independent over $\mathbb{Q}$. We are interested in the distribution of a half-infinite geodesic $\mathcal{L}(t)$, $t \geqslant 0$, with starting point $\mathcal{L}(0)$ and direction $\mathbf{v}^*$ in the associated polycube translation 3-manifold $\mathcal{M}$.

We have a simple equivalence principle relating half-infinite Kronecker sequences and half-infinite geodesics with Kronecker direction.

Theorem 1. *Suppose that $\mathcal{P}$ is a finite polysquare translation surface, and that $\mathcal{M} = \mathcal{P} \times [0, 1)$ denotes the associated polycube translation 3-manifold which is the cartesian product of $\mathcal{P}$ and the unit torus $[0, 1)$. Suppose also that $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ is a Kronecker vector, so that $\mathbf{v}^* = (v_1, v_2, 1) \in \mathbb{R}^3$ is a Kronecker direction. Then the following two statements are equivalent:*

(i) *Every half-infinite Kronecker sequence $\mathbf{v}_0 + j\mathbf{v}$, $j = 0, 1, 2, 3, \ldots,$ on $\mathcal{P}$ is uniformly distributed.*

(ii) *Every half-infinite geodesic $\mathcal{L}(t)$, $t \geqslant 0$, with direction $\mathbf{v}^*$ in $\mathcal{M}$ is uniformly distributed.*

Sketch of proof. ((ii) $\Rightarrow$ (i)) The argument here is rather simple. Suppose that $\mathcal{P}$ has s atomic squares. To establish (i), let S be a convex set in an atomic square of $\mathcal{P}$. Consider the first J terms of the Kronecker sequence, given by $\mathbf{v}_0 + j\mathbf{v}$, $j = 0, 1, \ldots, J - 1$. The number of terms of this finite sequence in S is given by

$$|\{j = 0, 1, \ldots, J - 1 : \mathbf{v}_0 + j\mathbf{v} \in S\}|,$$

with corresponding expectation given by

$$\frac{J\lambda_2(S)}{s},$$

where λ_2 denotes 2-dimensional Lebesgue measure. To establish uniformity of the Kronecker sequence on $\mathcal{P}$, we need to show that

$$|\{j=0,1,\ldots,J-1:\mathbf{v}_0+j\mathbf{v}\in S\}|\bigg/\frac{J\lambda_2(S)}{s}\to 1\quad\text{as } J\to\infty. \tag{2.1}$$

Let $\mathcal{L}(t)$, $t\geqslant 0$, be a half-infinite geodesic with starting point $\mathcal{L}(0)=(\mathbf{v}_0,0)$ and direction $\mathbf{v}^*$ on $\mathcal{M}$. Let $S^*\subset\mathcal{M}$ be obtained by sweeping the set $S\times\{0\}$ along by the vector $\mathbf{v}^*$, so that $\lambda_3(S^*)=\lambda_2(S)$, where λ_3 denotes 3-dimensional Lebesgue measure. Consider a finite geodesic segment $\mathcal{L}(t)$, $0\leqslant t\leqslant T=J(v_1^2+v_2^2+1)^{1/2}$. Then the total length of the parts of this geodesic segment in S^* is given by

$$\begin{aligned}&|\{0\leqslant t\leqslant T:\mathcal{L}(t)\in S^*\}|\\&\quad=(v_1^2+v_2^2+1)^{1/2}|\{j=0,1,\ldots,J-1:\mathbf{v}_0+j\mathbf{v}\in S\}|,\end{aligned} \tag{2.2}$$

with corresponding expectation given by

$$\frac{T\lambda_3(S^*)}{s}=\frac{J(v_1^2+v_2^2+1)^{1/2}\lambda_2(S)}{s}. \tag{2.3}$$

The set S^* is a union of finitely many convex sets in atomic cubes of $\mathcal{M}$. Suppose that (ii) holds. Uniformity of the half-infinite geodesic in $\mathcal{M}$ then implies that

$$|\{0\leqslant t\leqslant T:\mathcal{L}(t)\in S^*\}|\bigg/\frac{T\lambda_3(S^*)}{s}\to 1\quad\text{as } T\to\infty. \tag{2.4}$$

It is clear that (2.1) follows immediately on combining (2.2)–(2.4).

((i) $\Rightarrow$ (ii)) In [3, Section 3.4.1] or [1, Section 3.6], a result concerning uniformity of a half-infinite geodesic is deduced from a result concerning uniformity of a sequence, by an application of the Koksma inequality. Here we apply an analogous Koksma inequality type argument. ☐

The Gutkin–Veech theorem gives uniformity to any half-infinite geodesic with Kronecker direction on a finite polysquare translation surface. As a simple application of Theorem 1, we establish the analogous result for half-infinite Kronecker sequences.

Theorem 2. *Any half-infinite Kronecker sequence* $\mathbf{v}_0+j\mathbf{v}$*,* $j=0,1,2,3,\ldots,$ *on a finite polysquare translation surface* $\mathcal{P}$ *is uniformly distributed.*

Proof. Let $\mathcal{M}=\mathcal{P}\times[0,1)$ denote the associated polycube translation 3-manifold. In view of Theorem 1, it suffices to show that every half-infinite geodesic $\mathcal{L}(t)$, $t\geqslant 0$, with Kronecker direction in $\mathcal{M}$ is uniformly distributed. This latter condition is the conclusion of Theorem 3. ☐

The following result is [2, Theorem 3].

Theorem 3. *Suppose that a polycube translation* 3*-manifold* $\mathcal{M}$ *is the cartesian product of a finite polysquare translation surface* $\mathcal{P}$ *and the unit torus* $[0,1)$*. Then any half-infinite geodesic in* $\mathcal{M}$ *with a Kronecker direction* $\mathbf{v}^*\in\mathbb{R}^3$ *is uniformly distributed unless it hits a singularity.*

3. Stepping up principle

Theorem 2, concerning the distribution of half-infinite Kronecker sequences on $\mathcal{P}$, can be interpreted as a step-up from the Gutkin–Veech theorem, concerning the distribution of half-infinite geodesics with Kronecker direction on $\mathcal{P}$. In view of the equivalence given by Theorem 1, the stepping-up result is Theorem 3, which establishes the uniformity of a half-infinite geodesic with Kronecker direction on the associated polycube translation 3-manifold $\mathcal{M}$ from the uniformity of a half-infinite geodesic with Kronceker direction on the finite polysquare translation surface $\mathcal{P}$. This represents a step-up in dimension in some restricted way.

In this section, we expand on this idea. We establish the following results which, for simplicity, we state only in 2 and 3 dimensions. There are analogues in d and $d+1$ dimensions for every integer $d \geqslant 3$.

The discrete versions of these results concern the distribution of half-infinite Kronecker sequences on a finite polysquare translation surface $\mathcal{P}$ and the analogous problem in the associated polycube translation 3-manifold $\mathcal{M}$ which is the cartesian product of $\mathcal{P}$ and the unit torus $[0,1)$.

Theorem 4. *Suppose that $\mathcal{P}$ is a finite polysquare translation surface, and that $\mathcal{M}=\mathcal{P}\times[0,1)$ denotes the associated polycube translation 3-manifold which is the cartesian product of $\mathcal{P}$ and the unit torus $[0,1)$. Suppose also that $\mathbf{v}\in\mathbb{R}^2$ is the step vector of a Kronecker sequence on $\mathcal{P}$. Then the following statements are equivalent:*

(i) *Every half-infinite Kronecker sequence with step vector $\mathbf{v}$ on $\mathcal{P}$ is uniformly distributed.*

(ii) *For any $w\in\mathbb{R}$ such that $\mathbf{w}=(\mathbf{v},w)\in\mathbb{R}^3$ is the step vector of a Kronecker sequence on $\mathcal{M}$, every half-infinite Kronecker sequence with step vector $\mathbf{w}$ in $\mathcal{M}$ is uniformly distributed.*

Theorem 5. *Under the hypotheses of Theorem 4, the following statements are equivalent:*

(i) *The $\mathbf{v}$-shift on $\mathcal{P}$ is ergodic.*

(ii) *For any $w\in\mathbb{R}$ such that $\mathbf{w}=(\mathbf{v},w)\in\mathbb{R}^3$ is the step vector of a Kronecker sequence on $\mathcal{M}$, the $\mathbf{w}$-shift in $\mathcal{M}$ is ergodic.*

The continuous versions of these results concern the distribution of half-infinite geodesics with Kronecker direction on a finite polysquare translation surface $\mathcal{P}$ and the analogous problem in the associated polycube translation 3-manifold $\mathcal{M}$.

Theorem 6. *Suppose that $\mathcal{P}$ is a finite polysquare translation surface, and that $\mathcal{M}=\mathcal{P}\times[0,1)$ denotes the associated polycube translation 3-manifold which is the cartesian product of $\mathcal{P}$ and the unit torus $[0,1)$. Suppose also that $\mathbf{v}\in\mathbb{R}^2$ is a Kronecker direction. Then the following statements are equivalent:*

(i) *Every half-infinite geodesic with direction $\mathbf{v}$ on $\mathcal{P}$ is uniformly distributed.*

(ii) *For any $w\in\mathbb{R}$ such that $\mathbf{w}=(\mathbf{v},w)\in\mathbb{R}^3$ is a Kronecker direction, every half-infinite geodesic with direction $\mathbf{w}$ in $\mathcal{M}$ is uniformly distributed.*

Theorem 7. *Under the hypotheses of Theorem 4, the following statements are equivalent:*

(i) *Geodesic flow with direction $\mathbf{v}$ on $\mathcal{P}$ is ergodic.*

(ii) *For any $w\in\mathbb{R}$ such that $\mathbf{w}=(\mathbf{v},w)\in\mathbb{R}^3$ is a Kronecker direction, geodesic flow with direction $\mathbf{w}$ in $\mathcal{M}$ is ergodic.*

Remark. For Theorems 6 and 7, we already know that (i) and (ii) hold, in view of the Gutkin–Veech theorem and Theorem 3, so only the analogues in d and $d+1$ dimensions for integers $d\geqslant 3$ are new.

We concentrate our efforts on establishing Theorem 4. That (ii) implies (i) is almost trivial, by projection from $\mathcal{M}$ to $\mathcal{P}$. The converse is considerably harder.

Let $\mathbf{w}=(v_1,v_2,w)\in\mathbb{R}^3$, where $v_1,v_2,w,1$ are linearly independent over $\mathbb{Q}$, and consider the $\mathbf{w}$-shift $\mathbf{T}^*_{\mathbf{w}}=\mathbf{T}^*_{\mathbf{w}}(\mathcal{M})$ of geodesic flow in direction $\mathbf{w}$ in $\mathcal{M}=\mathcal{P}\times[0,1)$. We can consider two projections of $\mathbf{T}^*_{\mathbf{w}}$.

On the one hand, we can project the $\mathbf{w}$-shift $\mathbf{T}^*_{\mathbf{w}}$ to the unit torus $[0,1)^3$, simply by taking every coordinate modulo 1, leading to the $\mathbf{w}$-shift $\mathbf{T}_{\mathbf{w}}=\mathbf{T}_{\mathbf{w}}([0,1)^3)$ in the unit torus $[0,1)^3$ which is ergodic.

On the other hand, we can project the $\mathbf{w}$-shift $\mathbf{T}^*_{\mathbf{w}}$ to the polysquare translation surface $\mathcal{P}$, simply by dropping reference to the 3-rd coordinates throughout, leading to the $\mathbf{v}$-shift $\mathbf{T}^*_{\mathbf{v}} = \mathbf{T}^*_{\mathbf{v}}(\mathcal{P})$ on $\mathcal{P}$.

Lemma 3.1. *Suppose that the condition (i) in Theorem 4 holds. Then the* $\mathbf{v}$*-shift* $\mathbf{T}^*_{\mathbf{v}} = \mathbf{T}^*_{\mathbf{v}}(\mathcal{P})$ *on* $\mathcal{P}$ *is ergodic.*

Sketch of proof. If a Kronecker sequence on $\mathcal{P}$ becomes undefined after finitely many terms, then the supporting geodesic $\mathcal{L}(t)$, $t \geqslant 0$, with starting point $\mathcal{L}(0) = \mathbf{v}_0$ and direction $\mathbf{v}$ on $\mathcal{P}$ hits a singular point of $\mathcal{P}$. The collection of singular points of $\mathcal{P}$ clearly has 2-dimensional Lebesgue measure 0. Thus the collection of starting points $\mathbf{v}_0$ that lead a Kronecker sequence to be undefined after finitely many terms also has 2-dimensional Lebesgue measure 0. Thus almost every point $\mathbf{v}_0 \in \mathcal{P}$ gives rise to a half-infinite Kronecker sequence. Thus the condition (i) in Theorem 4 guarantees that for almost every starting point $\mathbf{v}_0$, the half-infinite Kronecker sequence $\mathbf{v}_0 + j\mathbf{v}$, $j = 0, 1, 2, 3, \ldots$, on $\mathcal{P}$ is uniformly distributed. Hence the visiting density of the sequence in any Jordan measurable set A is given by $\lambda_2(A)/s$.

Suppose, on the contrary, that the $\mathbf{v}$-shift $\mathbf{T}^*_{\mathbf{v}}$ on $\mathcal{P}$ is not ergodic. Then there is a partition $\mathcal{P} = U_1 \cup U_2$, where the subsets $U_1, U_2 \subset \mathcal{P}$ are $\mathbf{T}^*_{\mathbf{v}}$-invariant, and satisfy $\lambda_2(U_1) > 0$ and $\lambda_2(U_2) > 0$. Moreover, since the projection of $\mathbf{T}^*_{\mathbf{v}}$ to the unit torus $[0,1)^2$, simply by taking every coordinate modulo 1, leads to the $\mathbf{v}$-shift $\mathbf{T}_{\mathbf{v}} = \mathbf{T}_{\mathbf{v}}([0,1)^2)$ on the unit torus $[0,1)^2$ which is ergodic, there is a decomposition $\mathcal{P} = M_1 \cup \ldots \cup M_k$ for some integer k satisfying $2 \leqslant k \leqslant s$, where s is the number of atomic squares in $\mathcal{P}$, such that for every $i = 1, \ldots, k$, the set M_i is $\mathbf{T}^*_{\mathbf{v}}$-invariant and does not contain a proper $\mathbf{T}^*_{\mathbf{v}}$-invariant subset, and $\lambda_2(M_i)$ is a positive integer. We say that the subsets $M_1, \ldots, M_k$ are *minimal*.

We can find a Jordan measurable subset $A \subset \mathcal{P}$ such that

$$\lambda_2(A \cap M_1) < \frac{1}{10} \quad \text{and} \quad \lambda_2(A \cap M_2) > \frac{\lambda_2(M_2)}{2}.$$

Next we apply the Birkhoff ergodic theorem to both M_1 and M_2. The restriction of the $\mathbf{v}$-shift $\mathbf{T}^*_{\mathbf{v}}$ to M_1 and to M_2 are both ergodic. In each case, the simplest form of the ergodic theorem says that the *time average* is equal to the *space average*. Then for almost every starting point $\mathbf{v}_0 \in M_1$, the visiting density of the Kronecker sequence $\mathbf{v}_0 + j\mathbf{v}$, $j = 0, 1, 2, 3, \ldots$, on $\mathcal{P}$ in the set A is equal to the relative measure

$$\frac{\lambda_2(A \cap M_1)}{\lambda_2(M_1)} < \frac{1}{10},$$

while for almost every starting point $\mathbf{v}_0 \in M_2$, the visiting density of the Kronecker sequence $\mathbf{v}_0 + j\mathbf{v}$, $j = 0, 1, 2, 3, \ldots$, on $\mathcal{P}$ in the set A is equal to the relative measure

$$\frac{\lambda_2(A \cap M_2)}{\lambda_2(M_2)} > \frac{1}{2}.$$

Thus at least one of these is different from $\lambda_2(A)/s$, leading to a contradiction. □

Proof of Theorem 4. ((i) ⇒ (ii)) We need to prove that the $\mathbf{w}$-shift $\mathbf{T}^*_{\mathbf{w}}$ in $\mathcal{M}$ is ergodic. Suppose on the contrary that this is not the case. Then there exists a non-trivial partition $\mathcal{M} = \mathcal{W} \cup \mathcal{S}$ into a union of two $\mathbf{T}^*_{\mathbf{w}}$-invariant subsets $\mathcal{W}, \mathcal{S} \subset \mathcal{M}$, called White and Silver, say, each with integral measure and such that

$$1 \leqslant \lambda_3(\mathcal{W}), \lambda_3(\mathcal{S}) \leqslant s - 1,$$

where s is the number of atomic cubes in $\mathcal{M}$ and λ_3 denotes 3-dimensional Lebesgue measure.

Let $\mathcal{Y}_1,\ldots,\mathcal{Y}_s$ denote the atomic cubes of $\mathcal{M}$, and consider the projection of $\mathcal{M}$ to the unit torus $[0,1)^3$. Then for any point $P\in[0,1)^3$, there are precisely s points

$$P_1\in\mathcal{Y}_1,\quad \ldots,\quad P_s\in\mathcal{Y}_s$$

that have the same image P under this projection. Let

$$f_{\mathcal{W}}(P)=|\{P_1,\ldots,P_s\}\cap\mathcal{W}|\quad\text{and}\quad f_{\mathcal{S}}(P)=|\{P_1,\ldots,P_s\}\cap\mathcal{S}|.$$

Then $f_{\mathcal{W}}$ and $f_{\mathcal{S}}$ are positive integer valued functions defined on $[0,1)^3$ such that

$$f_{\mathcal{W}}(P)+f_{\mathcal{S}}(P)=s\quad\text{for almost every } P\in[0,1)^3.$$

The $\mathbf{w}$-shift $\mathbf{T}_{\mathbf{w}}$ in the unit torus $[0,1)^3$ is ergodic. Since the functions $f_{\mathcal{W}}$ and $f_{\mathcal{S}}$ are $\mathbf{T}_{\mathbf{w}}$-invariant, it follows from the Birkhoff ergodic theorem that they are constant almost everywhere in $[0,1)^3$. Then

$$f_{\mathcal{W}}+f_{\mathcal{S}}=s\quad\text{and}\quad 1\leqslant f_{\mathcal{W}},f_{\mathcal{S}}\leqslant s-1. \tag{3.1}$$

Let $\chi_{\mathcal{W}}$ denote the characteristic function of $\mathcal{W}$ in $\mathcal{M}$. For every $\mathbf{s}\in\mathcal{M}$, write $\mathbf{s}=(\mathbf{x},z)$, where $\mathbf{x}\in\mathcal{P}$ and $z\in[0,1)$. The well known Fubini theorem implies that

$$\int_{\mathcal{M}}\chi_{\mathcal{W}}(\mathbf{s})\,\mathrm{d}\mathbf{s}=\int_{\mathcal{P}}\left(\int_{[0,1)}\chi_{\mathcal{W}}(\mathbf{x};z)\,\mathrm{d}z\right)\mathrm{d}\lambda_2,$$

where the inner integral

$$\psi(\mathbf{x})=\int_{[0,1)}\chi_{\mathcal{W}}(\mathbf{x};z)\,\mathrm{d}z$$

is well defined in the sense of Lebesgue for almost every $\mathbf{x}\in\mathcal{P}$.

Consider now the projection of $\mathbf{T}^*_{\mathbf{w}}$ to the polysquare translation surface $\mathcal{P}$, resulting in the ergodic $\mathbf{v}$-shift $\mathbf{T}^*_{\mathbf{v}}$ on $\mathcal{P}$. Since the function $\psi(\mathbf{x})$ is $\mathbf{T}^*_{\mathbf{v}}$-invariant, it follows from ergodicity that it has constant value almost everywhere on $\mathcal{P}$. Thus it follows from (3.1) that

$$\frac{1}{s}\leqslant\psi=\frac{f_{\mathcal{W}}}{s}\leqslant\frac{s-1}{s} \tag{3.2}$$

almost everywhere on $\mathcal{P}$.

Since the invariant subset $\mathcal{W}\subset\mathcal{M}$ is measurable, it follows that for every $\varepsilon_1>0$, there exists a finite set of 3-dimensional axis-parallel rectangular boxes such that their union $\mathcal{B}=\mathcal{B}(\mathcal{W};\varepsilon_1)$ has the property that the symmetric difference

$$\mathcal{W}\mathbin{\triangle}\mathcal{B}=(\mathcal{W}\setminus\mathcal{B})\cup(\mathcal{B}\setminus\mathcal{W})$$

has measure

$$\lambda_3(\mathcal{W}\mathbin{\triangle}\mathcal{B})<\varepsilon_1. \tag{3.3}$$

We need the following simple technical result.

Let $\chi_{\mathcal{B}}$ denote the characteristic function of $\mathcal{B}$ in $\mathcal{M}$.

Lemma 3.2. *Let $\mathcal{B}$ be a finite union of axis-parallel rectangular boxes satisfying* (3.3)*. For every $\varepsilon_2>0$, there exists $\varepsilon_3>0$ and a finite set of disjoint axis-parallel rectangular boxes such that their union $\mathcal{B}^*=\mathcal{B}^*(\mathcal{B};\varepsilon_2;\varepsilon_3)$ satisfies the following conditions:*

(i) *The measure $\lambda_3(\mathcal{M}\setminus\mathcal{B}^*)<\varepsilon_2$.*

(ii) *We have $\chi_{\mathcal{B}}(\mathbf{s}')=\chi_{\mathcal{B}}(\mathbf{s}'')$ if the points $\mathbf{s}',\mathbf{s}''\in\mathcal{M}$ belong to the same axis-parallel rectangular box in the disjoint union $\mathcal{B}^*$.*

(iii) *The side lengths of each axis-parallel rectangular box in the disjoint union $\mathcal{B}^*$ is greater than ε_3.*

(iv) *Let s denote the number of atomic squares of $\mathcal{P}$. Then*

$$\lambda_2\left(\left\{\mathbf{x}\in\mathcal{P}:\lambda_1(\{z\in[0,1):(\mathbf{x},z)\in\mathcal{B}^*\})>1-\varepsilon_2^{1/2}\right\}\right)>s-\varepsilon_2^{1/2}.$$

Proof. We assume, without loss of generality, that every axis-parallel rectangular box in the finite union $\mathcal{B}$ lies in the interior of an atomic cube of $\mathcal{M}$, as illustrated in the picture on the left in Figure 1, which only shows 2 of the 3 dimensions. For each face of each axis-parallel rectangular box in an atomic cube of $\mathcal{M}$, we extend it to an axis-parallel *special* square face of area 1 in the same atomic cube, as illustrated by the dashed lines in the picture on the right in Figure 1. We repeat this process for every axis-parallel rectangular box in the finite union $\mathcal{B}$. Suppose that $\mathscr{N}=\mathscr{N}(\mathcal{B})$ denotes the total number of distinct axis-parallel rectangular boxes in the union $\mathcal{B}$. Then there are at most $2\mathscr{N}$ such distinct special square faces in each of the 3 axis-parallel directions. It follows that there exists a number $\varepsilon_4=\varepsilon_4(\mathcal{B})>0$ such that any special square face in an atomic cube has distance at least ε_4 from any other parallel special square face in the atomic cube or from any parallel boundary square face of the atomic cube. We now remove every point in $\mathcal{M}$ that lies a distance less than $\varepsilon_5>0$ from some special square face in the atomic cube that contains that point. Then the set of such points in $\mathcal{M}$ that are removed has measure at most $12\mathscr{N}\varepsilon_5$, and is represented by the regions shaded in light gray in the picture on the right in Figure 1.

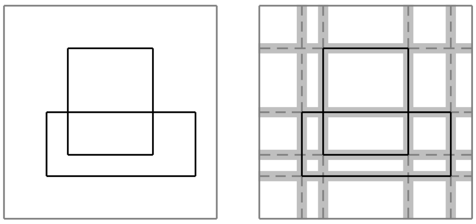

Figure 1: idea behind the construction of the set $\mathcal{B}^*$

Let $\mathcal{B}^*$ denote the remainder of $\mathcal{M}$ after these points are removed. Then $\mathcal{B}^*$ is clearly a finite union of disjoint axis-parallel rectangular boxes in $\mathcal{M}$, where each box is either contained in $\mathcal{B}$ or disjoint from $\mathcal{B}$, so that the condition (ii) is satisfied. Suppose now that $\varepsilon_5>0$ is chosen to satisfy

$$12\mathscr{N}\varepsilon_5<\varepsilon_2\quad\text{and}\quad 2\varepsilon_5<\varepsilon_4. \tag{3.4}$$

Then the first condition in (3.4) ensures that the condition (i) is satisfied, while the second condition in (3.4) ensures that the side lengths of each axis-parallel rectangular box in the union $\mathcal{B}^*$ is greater than $\varepsilon_3=\varepsilon_4-2\varepsilon_5$, so that the condition (iii) is satisfied.

To establish the condition (iv), note that it follows from the condition (i) and the Fubini theorem that

$$\int_{\mathcal{P}}\lambda_1(\{z\in[0,1):(\mathbf{x},z)\not\in\mathcal{B}^*\})\,\mathrm{d}\mathbf{x}<\varepsilon_2. \tag{3.5}$$

Let

$$\mathcal{E}=\left\{\mathbf{x}\in\mathcal{P}:\lambda_1(\{z\in[0,1):(\mathbf{x},z)\not\in\mathcal{B}^*\})\geqslant\varepsilon_2^{1/2}\right\}.$$

Then it follows from (3.5) that $\lambda_2(\mathcal{E})\varepsilon_2^{1/2}<\varepsilon_2$, from which the condition (iv) follows immediately, since $\lambda_2(\mathcal{P})=s$. $\square$

For almost every $\mathbf{x}\in\mathcal{P}$, the set

$$U(\mathcal{W};\mathbf{x})=\{z\in[0,1):(\mathbf{x},z)\in\mathcal{W}\}$$

is measurable, and it follows from (3.2) that

$$\frac{1}{s} \leqslant \lambda_1(U(\mathcal{W};\mathbf{x})) = \frac{f_{\mathcal{W}}}{s} \leqslant \frac{s-1}{s}. \tag{3.6}$$

We consider Lebesgue measurable subsets $U_\sigma \subset [0,1)$, $\sigma = 1,\ldots,s$, of the unit torus $[0,1)$. In particular, we make the assumption that $0 < \lambda_1(U_1) < 1$, where λ_1 denotes 1-dimensional Lebesgue measure. Furthermore, for any real number $u \in \mathbb{R}$ and any $\sigma = 1,\ldots,s$, we consider the $(-u)$-translated copy of U_σ, given by

$$U_\sigma - u = \{\{z-u\} : z \in U_\sigma\}.$$

Let $\mathbf{x}_1,\ldots,\mathbf{x}_s \in \mathcal{P}$ be distinct points such that their images under projection modulo 1 to the unit torus $[0,1)^2$ coincide. We now apply the following variant of [2, Lemma 6.1] to the sets

$$U_1 = U(\mathcal{W};\mathbf{x}_1), \quad \ldots, \quad U_s = U(\mathcal{W};\mathbf{x}_s), \tag{3.7}$$

so that for every $\sigma = 1,\ldots,s$,

$$z \in U_\sigma \quad \text{if and only if} \quad (\mathbf{x}_\sigma, z) \in \mathcal{W}.$$

Lemma 3.3. *The set of values $u_0 \in [0,1)$ for which the inequalities*

$$\lambda_1(U_1 \triangle (U_\sigma - u_0)) \geqslant \frac{1}{32s^2}\lambda_1(U_1)(1-\lambda_1(U_1)), \quad \sigma = 1,\ldots,s, \tag{3.8}$$

hold simultaneously has Lebesgue measure at least $1/2$.

It then follows on combining (3.6)–(3.8) that the set of values $u_0 \in [0,1)$ for which the inequalities

$$\lambda_1(U_1 \triangle (U_\sigma - u_0)) \geqslant \frac{1}{32s^4}, \quad \sigma = 1,\ldots,s, \tag{3.9}$$

hold simultaneously has Lebesgue measure at least $1/2$. Let

$$\mathscr{S}(\mathcal{W};\mathbf{x}_1) = \{u_0 \in [0,1) : (3.9) \text{ holds for every } \sigma = 1,\ldots,s\} \tag{3.10}$$

denote this set of such values of u_0. Note that the condition (3.9) is equivalent to the condition

$$\lambda_1(\{z \in [0,1) : \chi_{\mathcal{W}}(\mathbf{x}_1,z)\chi_{\mathcal{W}}(\mathbf{x}_\sigma,\{z+u_0\}) = 0\}) \geqslant \frac{1}{32s^4}, \quad \sigma = 1,\ldots,s. \tag{3.11}$$

Let us revisit the disjoint union $\mathcal{B}^* = \mathcal{B}^*(\mathcal{B};\varepsilon_2;\varepsilon_3)$ of axis-parallel rectangular boxes. For each axis-parallel rectangular box in this union, we push each boundary face inwards by a distance ε_6, where the parameter $\varepsilon_6 > 0$, to be specified later, satisfies

$$0 < \varepsilon_6 < \frac{\varepsilon_3}{2},$$

where ε_3 is a lower bound of the side lengths of these axis-parallel rectangular boxes. Then the resulting smaller axis-parallel rectangular box has volume which is at least $(1-2\varepsilon_6/\varepsilon_3)^3$ times that of the original axis-parallel rectangular box. This means that if $\mathcal{B}^{**} = \mathcal{B}^{**}(\mathcal{B}^*;\varepsilon_6)$ is the disjoint union of these smaller axis-parallel rectangular boxes, then using Lemma 3.2(i) and writing

$$\varepsilon_7 = \varepsilon_2 + \frac{6s\varepsilon_6}{\varepsilon_3}, \tag{3.12}$$

we have

$$\begin{aligned}
\lambda_3(\mathcal{B}^{**}) &\geqslant \left(1-\frac{2\varepsilon_6}{\varepsilon_3}\right)^3 \lambda_3(\mathcal{B}^*) \geqslant \left(1-\frac{2\varepsilon_6}{\varepsilon_3}\right)^3 (s-\varepsilon_2) \\
&\geqslant \left(1-\frac{6\varepsilon_6}{\varepsilon_3}\right)(s-\varepsilon_2) \geqslant s-\varepsilon_2-\frac{6s\varepsilon_6}{\varepsilon_3} = s-\varepsilon_7. \qquad (3.13)
\end{aligned}$$

Remark. Note that the ε_6-neighbourhood of every point in $\mathcal{B}^{**}$ is contained in $\mathcal{B}^*$.

Observe that $\chi_{\mathcal{B}}(\mathbf{s}') = \chi_{\mathcal{B}}(\mathbf{s}'')$ if the points $\mathbf{s}', \mathbf{s}'' \in \mathcal{M}$ belong to the same axis-parallel rectangular box in the disjoint union $\mathcal{B}^{**}$, and that the side lengths of each axis-parallel rectangular box in the disjoint union $\mathcal{B}^{**}$ is greater than $\varepsilon_3 - 2\varepsilon_6$. Furthermore, analogous to Lemma 3.2(iv), we have the inequality

$$\lambda_2\left(\left\{\mathbf{x} \in \mathcal{P} : \lambda_1(\{z \in [0,1) : (\mathbf{x}, z) \in \mathcal{B}^{**}\}) > 1 - \varepsilon_7^{1/2}\right\}\right) > s - \varepsilon_7^{1/2}. \tag{3.14}$$

For any $\mathbf{x} \in \mathcal{P}$, let $\mathbf{x}_1, \ldots, \mathbf{x}_s \in \mathcal{P}$ be distinct points such that their images under projection modulo 1 to the unit torus $[0,1)^2$ coincides with the image of $\mathbf{x}$ under the same projection. Then it follows from (3.14) that

$$\lambda_2\left(\left\{\mathbf{x} \in \mathcal{P} : \lambda_1(\{z \in [0,1) : (\mathbf{x}_\sigma, z) \in \mathcal{B}^{**}\}) > 1 - \varepsilon_7^{1/2} \text{ for every } \sigma = 1, \ldots, s\right\}\right)$$
$$> s - s\varepsilon_7^{1/2}. \tag{3.15}$$

For every $u_0 \in [0,1)$, let

$$\mu(u_0) = \lambda_2(\{\mathbf{x}_1 \in \mathcal{P} : u_0 \in \mathscr{S}(\mathcal{W}; \mathbf{x}_1)\})$$

denote some *relevant multiplicity* of u_0. Then

$$\int_{[0,1)} \mu(u_0)\,\mathrm{d}u_0 = \int_{\mathcal{P}} \lambda_1(\mathscr{S}(\mathcal{W}; \mathbf{x}_1))\,\mathrm{d}\mathbf{x}_1 \geqslant \frac{s}{2},$$

in view of (3.10). This can be interpreted to say that the *average multiplicity* $\mu(u_0)$ of $u_0 \in [0,1)$ is at least $s/2$. Thus there exists a shift $u_0^* \in [0,1)$ such that

$$\mu(u_0^*) = \lambda_2(\{\mathbf{x}_1 \in \mathcal{P} : u_0^* \in \mathscr{S}(\mathcal{W}; \mathbf{x}_1)\}) \geqslant \frac{s}{2}.$$

It also follows from (3.15) that with the exception of at most $s\varepsilon_7^{1/2}$ part of $\mathbf{x} \in \mathcal{P}$, at least $1 - 2\varepsilon_7^{1/2}$ part of the real numbers $z \in [0,1)$ are such that with $\mathbf{x} = \mathbf{x}_1$,

$$(\mathbf{x}_\sigma, z) \in \mathcal{B}^{**}, \quad (\mathbf{x}_\sigma, \{z + u_0^*\}) \in \mathcal{B}^{**}, \quad \sigma = 1, \ldots, s.$$

In order to derive a contradiction, we need a density analogue of [2, Lemma 6.2]. Here $\|\beta\|$ denotes the distance of $\beta \in \mathbb{R}$ to the nearest integer.

Lemma 3.4. *Let $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$ be a Kronecker vector, and let $w \in \mathbb{R}$ be arbitrary such that $\mathbf{w} = (v_1, v_2, w) \in \mathbb{R}^3$ is a Kronecker vector. Let $\varepsilon_6 > 0$ be given. There exists a finite sequence*

$$1 \leqslant m_1(\varepsilon_6) < \ldots < m_k(\varepsilon_6)$$

of positive integers such that

$$\|m_j(\varepsilon_6)v_1\| < \varepsilon_6, \quad \|m_j(\varepsilon_6)v_2\| < \varepsilon_6, \quad j = 1, \ldots, k, \tag{3.16}$$

and the finite sequence $\{m_j(\varepsilon_6)w\}$, $j = 1, \ldots, k$, visits every subinterval of $[0,1)$ with length ε_6.

Remark. Note that the number $k = k(\mathbf{w}; \varepsilon_6)$ of terms of the finite sequence may depend on the choice of $\mathbf{w}$ and the value of ε_6.

Proof of Lemma 3.4. By the Kronecker density theorem, the sequence

$$j\mathbf{w} = j(v_1, v_2, w), \quad j = 1, 2, 3, \ldots,$$

modulo 1 is dense in the unit torus $[0,1)^3$. Let

$$m_1(\varepsilon_6), m_2(\varepsilon_6), m_3(\varepsilon_6), \ldots$$

be the infinite subsequence of $1, 2, 3, \ldots$ such that

$$\{m_j(\varepsilon_6)v_1\}, \{m_j(\varepsilon_6)v_2\} \in [0, \varepsilon_6) \cup (1 - \varepsilon_6, 1), \quad j = 1, 2, 3, \ldots.$$

Clearly (3.16) holds. Also, the subsequence $m_j(\varepsilon_6)\mathbf{w}$, $j = 1, 2, 3, \ldots$, modulo 1 is dense in $([0, \varepsilon_6) \cup (1 - \varepsilon_6, 1))^2 \times [0, 1) \subset [0, 1)^3$. This implies that the sequence $\{m_j(\varepsilon_6)w\}$, $j = 1, 2, 3, \ldots$, is dense in $[0, 1)$.

Next, let the integer n satisfy $n > 2/\varepsilon_6$. We now partition the unit torus $[0, 1)$ into n short intervals $I_1, \ldots, I_n$ of length $1/n$ in the standard way. Then for every $i = 1, \ldots, n$, there exists an integer k_i such that the finite sequence

$$\{m_j(\varepsilon_6)w\}, \quad j = 1, \ldots, k_i,$$

visits I_i. Let $k = \max\{k_1, \ldots, k_n\}$. Then the finite sequence

$$\{m_j(\varepsilon_6)w\}, \quad j = 1, \ldots, k, \tag{3.17}$$

visits every interval $I_1, \ldots, I_n$. Now, since $1/n < \varepsilon_6/2$, every subinterval of $[0, 1)$ with length ε_6 must contain at least one of the intervals $I_1, \ldots, I_n$, so is visited by the finite sequence (3.17). □

It follows that there exists $j_0 = j_0(u_0^*)$ satisfying $1 \leqslant j_0 \leqslant k$ such that

$$\|m_{j_0}(\varepsilon_6)v_1\| < \varepsilon_6, \quad \|m_{j_0}(\varepsilon_6)v_2\| < \varepsilon_6, \quad \|m_{j_0}(\varepsilon_6)w - u_0^*\| < \varepsilon_6. \tag{3.18}$$

Remark. For a fixed point $\mathbf{x} \in \mathcal{P}$, we can visualize the set $\{(\mathbf{x}, z) : z \in [0, 1)\}$ as a *circle* over the point $\mathbf{x}$, since $[0, 1)$ is the unit torus. For any point $(\mathbf{x}, z)$ on this circle, we have $\mathbf{T}^*_{\mathbf{w}}(\mathbf{x}, z) = (\mathbf{x} + \mathbf{v}, \{z + w\})$. It follows that the image of the circle under the transformation $\mathbf{T}^*_{\mathbf{w}}$ is a circle $\{(\mathbf{y}, z') : z' \in [0, 1)\}$ over the point $\mathbf{y} = \mathbf{x}+\mathbf{v}$. Clearly, this new circle is rotated from the original circle by a quantity w and its position on $\mathcal{P}$ is translated from the original position by a vector $\mathbf{v}$. This action is repeated multiple times when we apply the transformation $\mathbf{T}^*_{\mathbf{w}}$ successively. Our proof of Theorem 4((i)⇒(ii)) is based on a combination of this fact with Lemmas 3.2–3.4.

Let us continue the discussion prior to Lemma 3.4. For at least $s/2 - s\varepsilon_7^{1/2}$ part of the points $\mathbf{x} \in \mathcal{P}$, we have $u_0^* \in \mathscr{S}(\mathcal{W}; \mathbf{x})$, and that for at least $1 - 2\varepsilon_7^{1/2}$ part of the real numbers $z \in [0, 1)$, writing $\mathbf{x} = \mathbf{x}_1$, we have

$$(\mathbf{x}, z) \in \mathcal{B}^{**}, \quad (\mathbf{x}_\sigma, \{z + u_0^*\}) \in \mathcal{B}^{**}, \quad \sigma = 1, \ldots, s. \tag{3.19}$$

We say that the points in (3.19) form a *good* $(s + 1)$-tuple, in the sense that they all belong to $\mathcal{B}^{**}$.

Let

$$Q = (\mathbf{x}, z) = (\mathbf{x}_1, z), \quad Q_\sigma = (\mathbf{x}_\sigma, \{z + u_0^*\}), \quad \sigma = 1, \ldots, s,$$

be such a good $(s+1)$-tuple, and consider the new point $Q^* = (\mathbf{T}^*_{\mathbf{w}})^{m_{j_0}}(Q)$, obtained from Q by m_{j_0} successive applications of the transformation $\mathbf{T}^*_{\mathbf{w}}$, where j_0 is chosen so that the inequalities (3.18) hold. Since the subset $\mathcal{W} \subset \mathcal{M}$ is invariant under the transformation $\mathbf{T}^*_{\mathbf{w}}$, it follows that $\chi_{\mathcal{W}}(Q) = \chi_{\mathcal{W}}(Q^*)$.

On the other hand, as observed in the Remark above, the image of the circle $\{(\mathbf{x}, z) : z \in [0, 1)\}$ under the transformation $(\mathbf{T}^*_{\mathbf{w}})^{m_{j_0}}$ is a circle $\{(\mathbf{y}, z') : z' \in [0, 1)\}$ with some particular $\mathbf{y} = \mathbf{x}_1 + m_{j_0}\mathbf{v} \in \mathcal{P}$. It then follows from (3.18) that the coordinates of $\mathbf{y}$ are ε_6-close to the corresponding coordinates of $\mathbf{x}_{\sigma_0}$ for a particular $\sigma_0 = 1, \ldots, s$, and the last coordinate of Q^* is ε_6-close to $\{z+u_0^*\}$. Since $Q_{\sigma_0} \in \mathcal{B}^{**}$, it follows from the Remark after (3.13) that Q_{σ_0} and Q^* are in the same axis-parallel rectangular box in the disjoint union $\mathcal{B}^*$, and so $\chi_{\mathcal{B}}(Q_{\sigma_0}) = \chi_{\mathcal{B}}(Q^*)$, in view of Lemma 3.2(ii).

Recall that $u_0^* \in \mathscr{S}(\mathcal{W}; \mathbf{x})$ for at least $s/2 - s\varepsilon_7^{1/2}$ part of the points $\mathbf{x} \in \mathcal{P}$. This and the condition (3.11) together imply that

$$\lambda_1(\{z \in [0, 1) : \chi_{\mathcal{W}}(\mathbf{x}_1, z)\chi_{\mathcal{W}}(\mathbf{x}_{\sigma_0}, \{z + u_0^*\}) = 0\}) \geqslant \frac{1}{32s^4},$$

and this is equivalent to

$$\lambda_1(\{z \in [0,1) : \chi_{\mathcal{W}}(Q) \neq \chi_{\mathcal{W}}(Q_{\sigma_0})\}) \geqslant \frac{1}{32s^4}. \tag{3.20}$$

Consider now the three relations

$$\chi_{\mathcal{W}}(Q) = \chi_{\mathcal{W}}(Q^*), \quad \chi_{\mathcal{B}}(Q_{\sigma_0}) = \chi_{\mathcal{B}}(Q^*), \quad \chi_{\mathcal{W}}(Q) \neq \chi_{\mathcal{W}}(Q_{\sigma_0}),$$

which clearly imply the two relations

$$\chi_{\mathcal{W}}(Q_{\sigma_0}) \neq \chi_{\mathcal{W}}(Q^*), \quad \chi_{\mathcal{B}}(Q_{\sigma_0}) = \chi_{\mathcal{B}}(Q^*).$$

It follows that

$$\chi_{\mathcal{W}}(Q_{\sigma_0}) \neq \chi_{\mathcal{B}}(Q_{\sigma_0}) \quad \text{or} \quad \chi_{\mathcal{W}}(Q^*) \neq \chi_{\mathcal{B}}(Q^*). \tag{3.21}$$

Intuitively speaking, (3.21) represents two *negligible* cases, with total measure less than ε_1, in view of (3.3), which contradict the substantial constant lower bound in (3.20) if $\varepsilon_1 > 0$ is sufficiently small. To make this precise, we need to study more closely the various parameters.

We have $u_0^* \in \mathscr{S}(\mathcal{W};\mathbf{x})$ for at least $s/2 - s\varepsilon_7^{1/2}$ part of the points $\mathbf{x} \in \mathcal{P}$. Using (3.20), we deduce that for at least

$$\frac{1}{32s^4} - 2\varepsilon_7^{1/2} \tag{3.22}$$

part of the real numbers $z \in [0,1)$, the points

$$Q_{\sigma_0} = (\mathbf{x}_{\sigma_0}, \{z + u_0^*\}) \quad \text{and} \quad Q^* = (\mathbf{T}_{\mathbf{w}}^*)^{m_{j_0}}(Q) = (\mathbf{T}_{\mathbf{w}}^*)^{m_{j_0}}(\mathbf{x}_1, z) \tag{3.23}$$

exhibit the property (3.21). It follows from (3.22) that the 3-dimensional Lebesgue measure of the points $Q = (\mathbf{x}_1, z) \in \mathcal{M}$ such that the points (3.23) exhibit the property (3.21) is at least

$$\left(\frac{s}{2} - s\varepsilon_7^{1/2}\right)\left(\frac{1}{32s^4} - 2\varepsilon_7^{1/2}\right), \tag{3.24}$$

where ε_7 is given by (3.12).

On the other hand, the property (3.21) is *exceptional*, and (3.3) implies that the quantity in (3.24) is less than $2\varepsilon_1$. We emphasize the fact that the choice of the parameter ε_1 is independent of the choices of the other parameters $\varepsilon_2, \varepsilon_3, \varepsilon_6$. Thus we can make ε_7 in (3.24) arbitrarily small independently of the fixed value of ε_1. It is therefore easy to specify the parameters $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_6$ so that the value of the quantity in (3.24) is greater than $2\varepsilon_1$, leading to a contradiction.

The contradiction establishes the ergodicity of the $\mathbf{w}$-shift in $\mathcal{M}$.

The last step is to extend ergodicity of the $\mathbf{w}$-shift in $\mathcal{M}$ to unique ergodicity by using the standard argument in functional analysis. This is possible, since the projection of $\mathcal{M}$ to the unit torus $[0,1)^3$ leads to unique ergodicity there. □

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