

ELEMENTARY MATHEMATICS

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Chapter 10

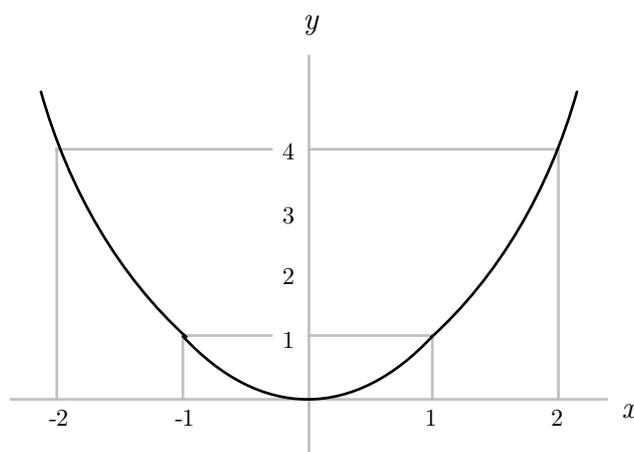
FUNCTIONS AND LINES

10.1. Functions and Graphs

We shall be concerned with real valued functions of a real variable x . In other words, we study functions of the form $f(x)$, where $x \in \mathbb{R}$ and $f(x)$ is real valued.

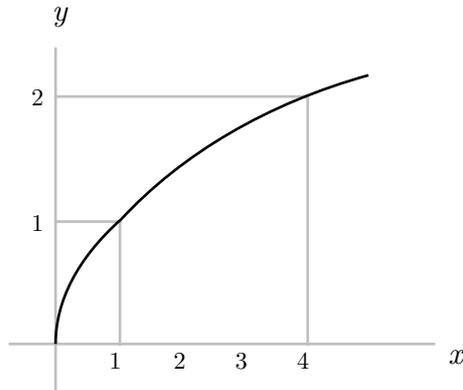
A convenient way to study and understand the properties of a function is to draw its graph. To do so, we make use of the xy -plane, and denote the values $f(x)$ by using the y -axis. Then the graph of the function consists of all points $(x, f(x))$ for which the function is defined.

EXAMPLE 10.1.1. Consider the function $f(x) = x^2$. This function is defined for all real values of x . For every $x \in \mathbb{R}$, the value $f(x)$ is real. We have $f(0) = 0$, $f(-1) = f(1) = 1$ and $f(-2) = f(2) = 4$. The graph of this function is given below.



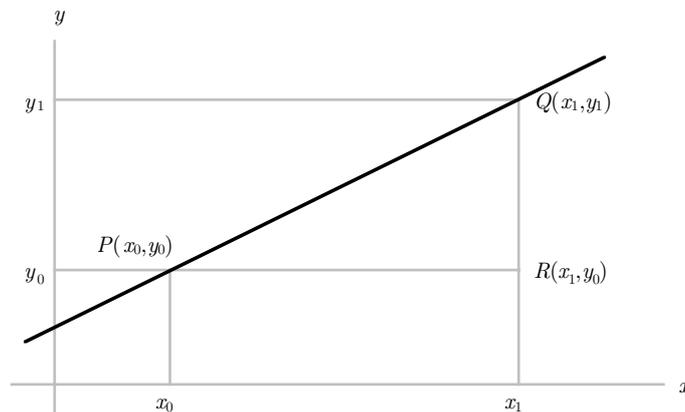
† This chapter was written at Macquarie University in 1999.

EXAMPLE 10.1.2. Consider the function $f(x) = \sqrt{x}$. This function is defined for all non-negative real values of x but not defined for any negative values of x . For every non-negative $x \in \mathbb{R}$, the value $f(x)$ is real. We have $f(0) = 0$, $f(1) = 1$, $f(2) = \sqrt{2}$, $f(3) = \sqrt{3}$ and $f(4) = 2$. The graph of this function is given below.



10.2. Lines on the Plane

In this section, we shall study the problem of lines and their graphs. Recall first of all that a line is determined if we know two of its points. Suppose that $P(x_0, y_0)$ and $Q(x_1, y_1)$ are two points on the xy -plane as shown in the picture below.



We can consider the triangle PQR formed by P and Q as well as the point $R(x_1, y_0)$. The length of the vertical side of this triangle is given by $y_1 - y_0$, while the length of the horizontal side of this triangle is given by $x_1 - x_0$. The ratio

$$\frac{y_1 - y_0}{x_1 - x_0}$$

is called the slope of the line through the points P and Q .

EXAMPLE 10.2.1. The line through the points $P(1, 2)$ and $Q(3, 6)$ has slope

$$\frac{6 - 2}{3 - 1} = 2.$$

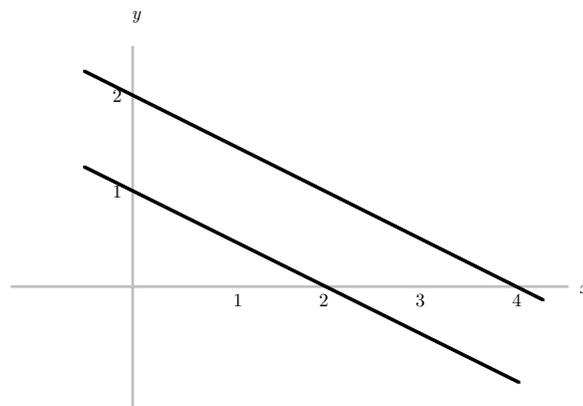
EXAMPLE 10.2.2. The line through the points $(4, 0)$ and $(0, 2)$ has slope

$$\frac{2 - 0}{0 - 4} = -\frac{1}{2}.$$

The line through the points $(2, 0)$ and $(0, 1)$ has slope

$$\frac{1 - 0}{0 - 2} = -\frac{1}{2}.$$

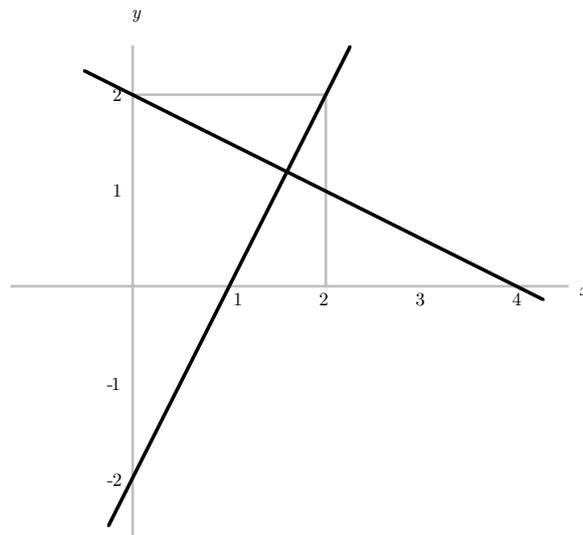
Note that the slope is negative in both cases; see the picture below.



It is clear that the two lines are parallel to each other. Note that their slopes are equal.

PARALLEL LINES. *Two lines on the plane are parallel to each other if and only if their slopes are equal.*

EXAMPLE 10.2.3. The line through the points $(4, 0)$ and $(0, 2)$ has slope $-1/2$. It is easy to check that the line through $(1, 0)$ and $(2, 2)$ has slope 2. We have the following picture.



It is clear that the two lines are perpendicular to each other. Note that the product of their slopes is equal to -1 .

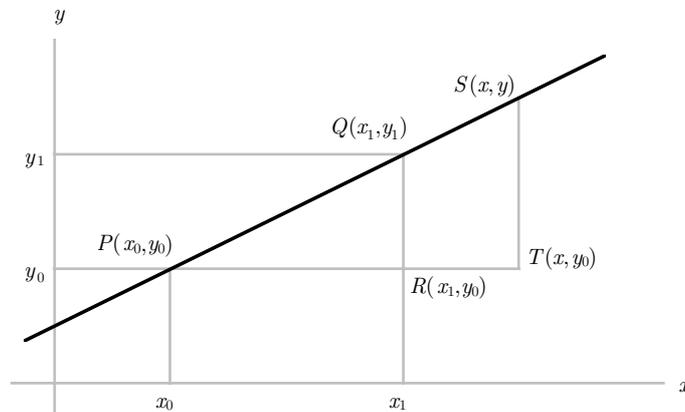
PERPENDICULAR LINES. Two lines on the plane are perpendicular to each other if and only if the product of their slopes is equal to -1 .

REMARK. The two results above are in fact not quite correct. For example, according to our definition, the line through the points $(1, 0)$ and $(1, 1)$ has slope

$$\frac{1 - 0}{1 - 1} = \frac{1}{0}$$

which is undefined. This represents a vertical line, parallel to the y -axis. On the other hand, a horizontal line, parallel to the x -axis, has slope 0 (why?). However, in these special cases, we do not need to use the two results above.

We now consider the problem of finding the equation of the line through two points $P(x_0, y_0)$ and $Q(x_1, y_1)$. Consider the picture below.



Suppose that (x, y) is a typical point on the line joining (x_0, y_0) and (x_1, y_1) . Note that the triangles PQR and PST are similar, so that

$$\frac{TS}{PT} = \frac{RQ}{PR}.$$

In other words, we must have

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0} = m,$$

where

$$m = \frac{y_1 - y_0}{x_1 - x_0}$$

is the slope of the line joining (x_0, y_0) and (x_1, y_1) . It follows that the equation of the line is given by

$$\frac{y - y_0}{x - x_0} = m.$$

This can be written in the form

$$y = mx + c, \tag{1}$$

where

$$c = y_0 - mx_0.$$

Note that the y -axis is the line $x = 0$. If we substitute this into (1), then we obtain $y = c$. This shows that the line (1) intersects the y -axis at the point $(0, c)$. This point is called the y -intercept of the line. On the other hand, the x -intercept of the line is the point of intersection with the x -axis. Note that the x -axis is the line $y = 0$. If we substitute this into (1), then we obtain $x = -c/m$. This shows that the line (1) intersects the x -axis at the point $(-c/m, 0)$.

REMARK. Again, our argument here fails if the line in question is a vertical line, parallel to the y -axis, or a horizontal line, parallel to the x -axis.

EXAMPLE 10.2.4. The line joining the points $(1, 0)$ and $(2, 2)$ is given by

$$\frac{y - 0}{x - 1} = \frac{2 - 0}{2 - 1} = 2.$$

This can be written in the form $y = 2x - 2$. Note that this line intersects the y -axis at the point $(0, -2)$ and the x -axis at the point $(1, 0)$. See the picture in Example 10.2.3.

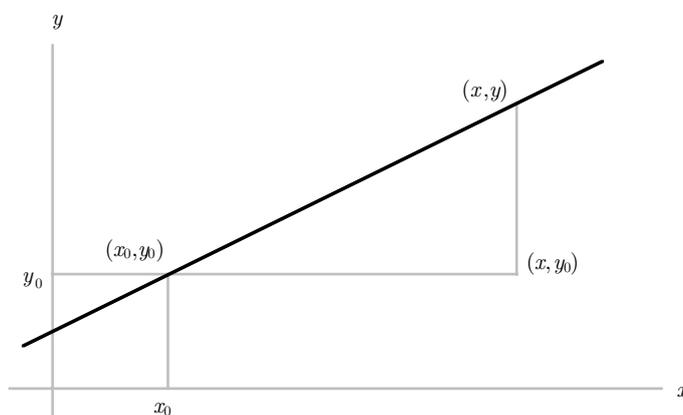
EXAMPLE 10.2.5. The line joining the points $(1, 1)$ and $(3, 1)$ is given by

$$\frac{y - 1}{x - 1} = \frac{1 - 1}{3 - 1} = 0.$$

This can be written in the form $y = 1$, with $m = 0$ and $c = 1$. Note that this line clearly intersects the y -axis at the point $(0, 1)$ but does not intersect the x -axis. Try to draw the graph of this line.

EXAMPLE 10.2.6. The line joining the points $(1, 1)$ and $(1, 3)$ is given by $x = 1$. This is an example of the exception to our technique. It is easy to see that the line is parallel to the y -axis. It intersects the x -axis at the point $(1, 0)$ but does not intersect the y -axis. Try to draw the graph of this line.

A line can also be determined if we know its slope and one of its points. Suppose that (x_0, y_0) is a given point, and that (x, y) is a typical point on the line through (x_0, y_0) with slope m . Consider the following picture.



Clearly we must have

$$\frac{y - y_0}{x - x_0} = m.$$

This again can be rewritten in the form $y = mx + c$, where $c = y_0 - mx_0$.

EXAMPLE 10.2.7. The line through the point $(1, 2)$ with slope 3 is given by

$$\frac{y - 2}{x - 1} = 3.$$

This can be rewritten in the form $y = 3x - 1$. Note that this line intersects the y -axis at the point $(0, -1)$ and the x -axis at the point $(1/3, 0)$.

EXAMPLE 10.2.8. The line $y = 4x - 2$ has slope 4. We now consider a line parallel to this and passing through the point $(1, 3)$. This parallel line must also have slope 4, and therefore has equation

$$\frac{y - 3}{x - 1} = 4.$$

This can be rewritten in the form $y = 4x - 1$. We consider next a line perpendicular to these two lines and passing through the point $(2, 1)$. This new line must have slope $-1/4$, and therefore has equation

$$\frac{y - 1}{x - 2} = -\frac{1}{4}.$$

This can be rewritten in the form $y = -\frac{1}{4}x + \frac{3}{2}$. Try to draw the graphs of these three lines, and find their x -intercepts and y -intercepts.

PROBLEMS FOR CHAPTER 10

- For each of the following functions, determine the value $f(x)$ for every integer x satisfying the inequalities $-3 \leq x \leq 3$, and draw a rough sketch of the graph of the function:
 - $f(x) = x^2 + 2x - 1$
 - $f(x) = 4 - x^2$
 - $f(x) = \sqrt{9 - x^2}$
- For each of the following, find the equation of the line through the given points:
 - $(1, 3)$ and $(2, 5)$
 - $(-1, 3)$ and $(2, 5)$
 - $(-3, 5)$ and $(2, 6)$
- For each of the following, find the equation of the line through the given points, and determine the x -intercept and y -intercept of this line:
 - $(2, 0)$ and $(3, 5)$
 - $(3, -2)$ and $(3, 1)$
 - $(5, -1)$ and $(-1, -1)$
- For each of the following, find the equation of the line with the given slope and passing through the given point, and determine the x -intercept and y -intercept of this line:
 - slope 3 and point $(3, 1)$
 - slope -2 and point $(-1, 4)$
 - slope 4 and point $(4, 2)$
- For each of the three lines in Question 3, find the equation of the perpendicular line which passes through the point $(-4, -4)$, as well as the intersection of the perpendicular line with the given line.
- For each of the three lines in Question 4, find the equation of the perpendicular line which passes through the point $(-1, -1)$, determine its point of intersection with the original line, and then find the equation of the line which passes through this intersection point and the origin $(0, 0)$.
- Consider the parabola $y = f(x)$ where $f(x) = x^2 + 3$.
 - Find the slope of the line joining the points (x_1, y_1) and (x_2, y_2) on this parabola, where $x_1 = 1$ and $x_2 = 3$.
 - Find the slope of the line joining the points (x_3, y_3) and (x_4, y_4) on this parabola, where $y_3 = 7$ and $y_4 = 19$ and where both x_3 and x_4 are negative.
 - Confirm that the slope of the line in part (a) is equal to $x_1 + x_2$, and that the slope of the line in part (b) is equal to $x_3 + x_4$.
 - Can you obtain the conclusion of part (c) without using the information about the specific values of x_1, x_2, y_3 and y_4 ?
- Let m_1 be the slope of some line, and let m_2 be the slope of a perpendicular line. Use trigonometry to explain why the product $m_1 m_2$ is equal to -1 , except when one line is horizontal and the other line is vertical.